Local Moduli of Sasaki-Einstein $\mathbb{Q}$-homology 7-spheres

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Sasakian-Einstein Geometry

Sasakian structures

Recall that a Sasakian structure on a contact manifold M^{2n+1} is a special type of contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with underlying almost CR structure $(\mathcal{D}, J)$ where

1. η is a contact form such that $\mathcal{D} = \ker \eta$,
2. ξ is its Reeb vector field,
3. $J = \Phi|_{\mathcal{D}}$ and $g = d\eta \circ (\mathbb{I} \times \Phi) + \eta \otimes \eta$ is a Riemannian metric.

$\mathcal{S}$ is **Sasakian** if in addition

4. ξ is a Killing vector field and
5. the almost CR structure is integrable, i.e. (D, J) is a CR structure.

Equivalently, the metric cone $(C(M), \bar{g}) = \left(\mathbb{R}_{>0} \times M,\ \mathrm{d}r^2 + r^2 g\right)$ over M is Kähler.

Sasakian structures

- The Reeb vector field ξ determines a foliation on M and the transverse space is also Kähler.
- When the foliation is regular the space is a smooth Kähler manifold. So we obtain

$$
\begin{array}{rl}
(\mathcal{C}(M), \bar{g}, \bar{\Phi}) \hookleftarrow & (M, \xi, \eta, g, \Phi) \\
& \qquad\quad \downarrow \\
& \quad\ (\mathcal{Z}, \omega, J).
\end{array}
$$

- When the foliation is quasi-regular the space is a Kähler orbifold.
- Examples of the latter are given by links of isolated hypersurface singularities.

Consider

1. The subvariety $Y := (F = 0) \subset \mathbb{C}^{n+1}$.
2. Suppose further that Y has only an isolated singularity at the origin.

Then the link

$$L_F = F^{-1}(0) \cap S^{2n+1}$$

of F is a smooth compact $(n-2)$-connected manifold of dimension $2n-1$.

Sasakian Geometry on links of isolated hypersurface singularities

In particular one can consider weighted homogeneous polynomials, that is polynomials $F : \mathbb{C}^{n+1} \to \mathbb{C}$ of degree d and weight $\mathbf{w} = (w_0, \ldots, w_n)$, that is, such that

$$F\left(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n\right) = \lambda^d F\left(z_0, \ldots, z_n\right),$$

In particular we have at our disposal a weighted $\mathbb{C}^*$ action on F that induces a S^1-action on the link

$$L_F(\mathbf{w}) = F^{-1}(0) \cap S^{2n+1}.$$

It follows that L_F admits a quasi-regular Sasaki structure $\mathcal{S} = \left(\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}}\right).$

Sasakian Geometry on links of isolated hypersurface singularities

The quotient space of the link L_F by this circle action is just the orbifold $X^{\mathrm{orb}} = (F^{-1}(0) - \{0\})/\mathbb{C}^*$. It has a natural Kähler structure:

$$
\begin{array}{ccc}
(F^{-1}(0) - \{0\}) & \hookrightarrow & \mathbb{C}^{n+1} \\
\downarrow & & \downarrow \\
L_F & \hookrightarrow & S^{2n+1} \\
\downarrow \pi & & \downarrow \\
X^{orb} & \hookrightarrow & \mathbb{CP}(\mathbf{w})
\end{array}
$$

Here $(F^{-1}(0) - \{0\})/\mathbb{C}^* = X^{orb}$.

Invertible polynomials and Einstein metrics on $\mathbb{Q}$-homology spheres

In [3] we establish the existence of Sasaki-Einstein metrics on links of invertible polynomials satisfying either

- The weights and the degree satisfy $\gcd(d, w_i) = 1$ for all $i = 0, \ldots 4$, which leads to singularities of cycle type:

$$z_0^{a_0} z_1 + z_1^{a_1} z_2 + z_2^{a_2} z_3 + z_3^{a_3} z_4 + z_4^{a_4} z_0$$

- The weights are subject to $(w_0, w_1, w_2, w_3, w_4) = (m_2 v_0, m_2 v_1, m_2 v_2, m_3 v_3, m_3 v_4)$ with $\gcd(m_2, m_3) = 1$ and $m_2 m_3 = d$, which leads to singularities that can described as the following iterated Thom-Sebastiani sums of chain, cycle type and Fermat singularities:

$$\begin{aligned} \text{Type I (Fermat-Cycle):}\ & z_0^{a_0} + z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4} \\ \text{Type II (Chain-Cycle):}\ & z_0^{a_0} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4} \\ \text{Type III (Cycle-Cycle):}\ & z_1 z_0^{a_0} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}. \end{aligned}$$

All these cases lead to all the Sasaki-Einstein links with rational homology of a 7-sphere of index 1 (i.e., $|\mathbf{w}| - d = 1$): this is due to a result given by Kollár and Johnson in the classification of weighted Fano 3-folds of index 1 (2003).

In particular, the corresponding affine varieties $F^{-1}(0)$ admits a Calabi-Yau structure and the corresponding orbifolds $Z_F = (F^{-1}(0) - 0)/\mathbb{C}^*$ are rational homology complex projective 3-spaces, that is,

$$H_*(Z_F, \mathbb{Q}) = H_*(\mathbb{CP}^3, \mathbb{Q}).$$

Local Moduli of Sasaki-Einstein $\mathbb{Q}$-rational homology 7-spheres

THEOREM: [2] Let Z_f be a quasi-smooth weighted hypersurface in $\mathbb{P}(\mathbf{w})$ corresponding to the weighted homogenous polynomial f of degree d and weight vector $\mathbf{w} = (w_0, \ldots, w_n)$. Then the complex orbifolds Z_f form a continuous family of complex dimension

$$h^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d)) - \sum_i h^0\left(\mathbb{P}(\mathbf{w}), \mathcal{O}\left(w_i\right)\right) + \dim \mathfrak{Aut}\left(Z_f\right).$$

Furthermore, if the index $I = |\mathbf{w}| - d > 0$ and Z_f admits a Kähler-Einstein metric for a generic f then it admits a $2\left[h^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d)) - \sum_i h^0\left(\mathbb{P}(\mathbf{w}), \mathcal{O}\left(w_i\right)\right) + \dim \mathfrak{Aut}\left(Z_f\right)\right]$ dimensional family of Kähler-Einstein metrics up to homothety.

THEOREM: [1] Let M be a smooth compact manifold and let $\mathcal{S} = (\xi, \eta, \Phi_t, g_t)$ be a family of quasi-regular Sasaki-Einstein structures on M induced by a continuous family of inequivalent complex orbifolds $\mathcal{Z}_t$ with Kähler-Einstein metrics. Then the metrics g_t are inequivalent as Einstein metrics.

The key is to consider all possible monomials $z_0^{x_0} z_1^{x_1} z_2^{x_2} z_3^{x_3} z_4^{x_4}$ of degree d which is equivalent to find all the solutions of the following Diophantine equation

$$w_0x_0 + w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 = d$$

with variables $x_i \in \mathbb{Z}_{\geq 0}$ and where at least one of them is nonzero and subject to certain arithmetic constraints given by the type of invertible polynomials under study.

Cycle polynomials

Consider

$$f = z_4 z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}$$

of degree d and $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$, such that $\gcd(d, w_i) = 1$ and $d = w_0 + w_1 + w_2 + w_3 + w_4 - 1$.

Clearly

$$a_0 w_0 + w_4 = d, a_1 w_1 + w_0 = d, a_2 w_2 + w_1 = d, a_3 w_3 + w_2 = d, a_4 w_4 + w_3 = d$$

Cycle polynomials

Since the link L_f is a $\mathbb{Q}$-homology sphere, from the Alexander polynomial one obtains $d = 1 + a_0 a_1 a_2 a_3 a_4$.

Using this, we can express each weight w_i as:

$$w_i = 1 - a_{i+1} + a_{i+1}a_{i+2} - a_{i+1}a_{i+2}a_{i+3} + a_{i+1}a_{i+2}a_{i+3}a_{i+4},$$

where the subscript is taking mod 5.

From $\gcd(d, w_i) = 1$, we conclude that two consecutive weights w_i and w_{i+1} are always co-primes.

Cylce polynomials

LEMMA: The Diophantine equation

$$w_0x_0 + w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 = d$$

associated to the problem above has exactly five solutions. These solutions are

$(a_0, 0, 0, 0, 1), (1, a_1, 0, 0, 0), (0, 1, a_2, 0, 0), (0, 0, 1, a_3, 0)$ and $(0, 0, 0, 1, a_4)$.

Thus,

$$H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d)) = Span\{z_0^{a_0}z_4, z_0z_1^{a_1}, z_1z_2^{a_2}, z_2z_3^{a_3}, z_3z_4^{a_4}\}.$$

LEMMA: The Diophantine equation

$$w_0x_0 + w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 = w_i$$

associated to the problem has a unique solution for each w_i, that is,

$$H^0\left(\mathbb{P}(\mathbf{w}), \mathcal{O}\left(w_i\right)\right) = \{z_i\}$$

for $i = 0 \ldots, 4$.

Cycle polynomials

Since

$$\mathbb{P}(\mathbf{w}) = \mathrm{Proj}(S(\mathbf{w})),$$

where $S(\mathbf{w}) = \bigoplus_d S^d(\mathbf{w}) = \mathbb{C}[z_0, z_1, z_2, z_3, z_4]$. The ring of polynomials $\mathbb{C}[z_0, z_1, z_2, z_3, z_4]$ is graded with grading defined by the weights $\mathbf{w} = (w_1, w_1, w_2, w_3, w_4)$. Since the group $G(\mathbf{w})$ of automorphisms of the graded ring $S(\mathbf{w})$ can be defined on generators by

$$\varphi_{\mathrm{w}} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} \alpha_0 z_0 \\ \alpha_1 z_1 \\ \alpha_2 z_2 \\ \alpha_3 z_3 \\ \alpha_4 z_4 \end{pmatrix}$$

where $\alpha_i \in \mathbb{C}^*$. The group $\mathfrak{G}_{\mathbf{w}}$ of complex automorphisms of $\mathbb{P}(\mathbf{w})$ is the projectivization of $G(\mathbf{w})$ which in this case is given by $\mathfrak{G}_{\mathbf{w}} = (\mathbb{C}^*)^4$.

Since $H^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d))$ is span by the monomials $z_0^{a_0} z_4, z_0 z_1^{a_1}, z_1 z_2^{a_2}, z_2 z_3^{a_3}, z_3 z_4^{a_4}$. It follows that the moduli of the orbifold Z_f, a $\mathbb{Q}$-homology projective 3-space, is given by

$$Span\{z_0^{a_0} z_4, z_0 z_1^{a_1}, z_1 z_2^{a_2}, z_2 z_3^{a_3}, z_3 z_4^{a_4}\}/G(\mathbf{w})$$

Cycle polynomials

Theorem

In particular $\mathbb{Q}$-homology 7-spheres given as links coming from polynomials

$$f = z_4 z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}$$

of degree d and $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$, such that $\gcd(d, w_i) = 1$ and $d = w_0 + w_1 + w_2 + w_3 + w_4 - 1$ do not admit inequivalent families of Sasaki-Einstein structures.

Thom-Sebastiani sums

For the following types:

Type I (Fermat-Cycle): $z_0^{a_0} + z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}$

Type II (Chain-Cycle): $z_0^{a_0} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}$

Type III (Cycle-Cycle): $z_1 z_0^{a_0} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}$

one can deduce that the weight vectors associated to these families of polynomials have the form:

$$\mathbf{w} = (w_0, w_1, w_2, w_3, w_4) = (m_3 v_0, m_3 v_1, m_2 v_2, m_2 v_3, m_2 v_4), \quad (1)$$

where $\gcd(m_2, m_3) = 1$ and, from well-formedness, $\gcd(v_0, v_1) = 1$ and $\gcd(v_i, v_j) = 1$ for $i \neq j$ with $i, j \in \{2, 3, 4\}$. Furthermore, assuming the link L_f correspond to a $\mathbb{Q}$-homology sphere the polynomial f has degree $d = m_2 m_3$ and index $I = |\mathbf{w}| - d = 1$,

Thom-Sebastiani sums

Notice that solving the corresponding Diophantine equation:

$$w_0 y_0 + w_1 y_1 + w_2 x_2 + w_3 x_3 + w_4 x_4 = d$$

with variables $y_i, x_i \in \mathbb{Z}_0^+$ and where at least one of them is nonzero,

is equivalent to solve the following two Diophantine equations

$$w_0 y_0 + w_1 y_1 = d$$

and

$$w_2 x_2 + w_3 x_3 + w_4 x_4 = d$$

Thom-Sebastiani sums

The group $G(\mathbf{w})$ of complex automorphisms of the graded ring $S(\mathbf{w})$ with Proj($S(\mathbf{w})$). Let $\alpha_i, \beta_1 \in \mathbb{C}^*$ and $\mathbb{A} \in GL(2, \mathbb{C})$, is given as follows

If f is a polynomial of type I, then $G(\mathbf{w})$ is given on generators by

$$\varphi_{\mathrm{w}} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} \mathbb{A} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \\ \alpha_2 z_2 \\ \alpha_3 z_3 \\ \alpha_4 z_4 \end{pmatrix}$$

and the moduli of the orbifold Z_f can be expressed by the quotient

$$Span\left\{z_1^{m_2}, z_0 z_1^{m_2-1}, \ldots, z_0^{m_2-1} z_1, z_0^{m_2}, z_4 z_2^{a_2}, z_2 z_3^{a_3}, z_3 z_4^{a_4}\right\} / G(\mathbf{w}).$$

Thom-Sebastiani sums

If f is a polynomial of type II and its associated weight vector $\mathbf{w}$ does not admit polynomial of type I, then $G(\mathbf{w})$ is given on generators by

$$\varphi_{\mathrm{w}} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} \alpha_0 z_0 \\ \alpha_1 z_1 + \beta_1 z_0^{v_1} \\ \alpha_2 z_2 \\ \alpha_3 z_3 \\ \alpha_4 z_4 \end{pmatrix}$$

and the moduli of the orbifold Z_f can be expressed by the quotient

$$Span\left\{ z_0^{m_2 - k v_1} z_1^k, z_4 z_2^{a_2}, z_2 z_3^{a_3}, z_3 z_4^{a_4},\ \text{where } 0 \leq k \leq \left\lfloor \frac{m_2}{v_1} \right\rfloor + 1 \right\} / G(\mathbf{w}).$$

Thom-Sebastiani sums

If f is a polynomial of type III and its associated weight vector $\mathbf{w}$ does not admit polynomial of type II, then $G(\mathbf{w})$ is given on generators by

$$\varphi_{\mathrm{w}} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} \alpha_0 z_0 \\ \alpha_1 z_1 \\ \alpha_2 z_2 \\ \alpha_3 z_3 \\ \alpha_4 z_4 \end{pmatrix}$$

and the moduli of the orbifold Z_f can be expressed by the quotient

$$Span\left\{ z_0^{a_0 - kv_1} z_1^{1+kv_0}, z_4 z_2^{a_2}, z_2 z_3^{a_3}, z_3 z_4^{a_4}, \text{ where } 0 \leq k \leq \left\lfloor \frac{m_2}{v_0 v_1} \right\rfloor + 1 \right\} / G(\mathbf{w}).$$

Thom-Sebastiani sums

Theorem For the following polynomials

Type I (Fermat-Cycle): $z_0^{a_0} + z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}$

Type II (Chain-Cycle): $z_0^{a_0} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}$

Type III (Cycle-Cycle): $z_1 z_0^{a_0} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}$

with weight vector
$\mathbf{w} = (w_0, w_1, w_2, w_3, w_4) = (m_3 v_0, m_3 v_1, m_2 v_2, m_2 v_3, m_2 v_4)$, and degree $d = m_2 m_3$. Then the real dimension of the local moduli of Sasaki-Einstein metrics of $\mathbb{Q}$-homology 7-spheres at L_f equals $2\left[\frac{m_2}{v_0 v_1} - \frac{1}{v_0} - \frac{1}{v_1} - 1\right]$.

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