Spray-Invariant Sets in Infinite-Dimensional Manifolds

Kaveh Eftekharinasb, Institute of Mathematics of NAS of Ukraine

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Motivation

- For infinite-dimensional manifolds (Hilbert, Banach, and Fréchet manifolds), we introduce a broader perspective on geodesic preservation than the rigid notion of totally geodesic submanifolds: subsets with the property that any geodesic of a spray starting in the set stays within it for its entire domain.
- Our framework includes singular spaces (e.g., stratified spaces), which are common in infinite dimensions. Such sets arise naturally even in simple settings (e.g., linear spaces equipped with flat sprays).
- Regularity matters! singular sets may show sensitive dependence, for example, on parametrization, whereas for differentiable submanifolds invariance is preserved under reparametrization.

Sprays

In $\mathbb{R}^n$, a symmetric bilinear form $\omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ can be entirely reconstructed from its associated quadratic form $a(t) = \omega(t, t)$ using the polarization identity:

$$\omega(t_1, t_2) = \frac{1}{2}\left(a(t_1 + t_2) - a(t_1) - a(t_2)\right).$$

Interestingly, this lifts beautifully to differential geometry! Ambrose, Palais, and Singer (1960) showed a similar natural bijection between:

1. Symmetric affine connections on a smooth manifold M. (analog of symmetric bilinear forms).
2. Sprays on the tangent bundle $\mathrm{T}(\mathrm{TM})$. (analog of quadratic forms).

Spray Geometry

In finite-dimensional Finsler theory, Finsler structures are defined by $F : \mathrm{TM} \to [0, \infty)$, which are

- Smooth on $\mathrm{TM} \backslash \{0\}$.
- Positively homogeneous and strongly convex on tangent spaces.

For our infinite-dimensional setting, we must give up the smoothness requirement. This is crucial because smoothness is too restrictive for many infinite-dimensional examples.
Unfortunately, this has a serious drawback: we cannot use standard Finsler geometric methods to obtain an exponential function and, hence, a spray.
We do not require the existence of a spray induced by a Finsler (or Riemannian) metric or compatibility with such a structure.

Michal–Bastiani Differentiability

Let $\varphi : U \to \mathsf{F}$ be a mapping, where U is open in E, and E, F are locally convex spaces.

- The directional derivative at x in direction h is

$$D\varphi_x(h) := \lim_{t\to 0} \frac{1}{t}(\varphi(x+th) - \varphi(x))$$

- φ is differentiable at x if $D\varphi_x(h)$ exists for all h.
- φ is continuously differentiable (C^1) if it's differentiable on U and the map $D\varphi\colon U \times \mathsf{E} \to \mathsf{F}, \quad (x,h) \mapsto D\varphi_x(h)$ is continuous.
- φ is a C^k-mapping ($k \in \mathbb{N} \cup \{\infty\}$) if it's continuous, all iterated directional derivatives $D^j\varphi_x(h_1,\ldots,h_j)$ exist for $j \leqslant k$, and all maps $D^j\varphi\colon U \times \mathsf{E}^j \to \mathsf{F}$ are continuous.

Defining Sprays

A C^r-mapping $V\colon \mathrm{TM} \to \mathrm{T(TM)}$ $(1 \leqslant r \leqslant k-2)$ is a second-order C^r-vector field if $\tau_* \circ V = \mathrm{Id}_{\mathrm{TM}}$. It is called symmetric if, in addition, $\tau_2 \circ V = \mathrm{Id}_{\mathrm{TM}}$, where $\tau_2\colon \mathrm{T(TM)} \to \mathrm{TM}$, $\tau\colon \mathrm{TM} \to \mathsf{M}$, M a manifold modeled on F.

Spray: A second-order symmetric C^r-vector field $\mathbf{S}\colon \mathrm{TM} \to \mathrm{T(TM)}$:

1. $\mathbf{S}(sv) = (L_{\mathrm{TM}})_*(s\mathbf{S}(v))$ for all $s \in \mathbb{R}$ and $v \in \mathrm{TM}$.

$$L_{\mathrm{TM}}\colon \mathrm{TM} \to \mathrm{TM}, \quad v \mapsto sv.$$

Consider a chart (U, φ) of M. The spray condition means that

$$(x, sv, sv, f(x, sv)) = (L_{\mathrm{TM}})_*(s\mathbf{S}(v))(x, v, sv, sf(x, v)) = (x, sv, sv, s^2 f(x, v))$$

$$f : U \times \mathsf{F} \to \mathsf{F}, \quad f(x, sv) = s^2 f(x, v)$$

indicating that for each fixed $x \in U$, the function $f(x, \cdot)$ must be *quadratic* in the velocity variables v.

Defining Geodesics via Sprays

A curve $g\colon I \subseteq \mathbb{R} \to M$ is a geodesic of a spray $\mathbf{S}$ if its canonical lifting $g'\colon I \to \mathrm{TM}$ is an integral curve of the spray $\mathbf{S}$.

This means the "acceleration" of the curve is determined by the spray:

$$g'' = \mathbf{S}(g')$$

Adjacent Cones

- The pseudo-distance of an element $x \in \mathsf{F}$ to a subset $S \subset \mathsf{F}$ with respect to a seminorm $\|\cdot\|_{\mathsf{F},n}$ is defined by

$$\mathrm{d}_{\mathsf{F},n}(x, S) := \inf\{ \|x - y\|_{\mathsf{F},n} \mid y \in S \}.$$

- Let $S \subset \mathrm{M}$, $s \in S$. A vector $v \in \mathrm{T}_s\mathrm{M}$ is called a first-order adjacent tangent vector to S at s if there exists a chart (U, φ) around s, s.t.:

$$\forall n \in \mathbb{N}, \quad \lim_{t \to 0^+} t^{-1} \mathrm{d}_{\mathsf{F},n}\Big(\varphi(s) + t\mathrm{D}\varphi(s)(v), \varphi(U \cap S)\Big) = 0.$$

  The set of all such v is denoted by $\mathrm{T}_s S$.

- Let $v \in \mathrm{T}_s S$. A vector $w \in \mathrm{T}_v(\mathrm{TM})$ is called a second-order adjacent tangent vector to S at s (with v as its associated first-order vector) if there exists a chart (U, φ) about s, s.t. $\mathrm{D}(\varphi_*)_v(w) = v$, and

$$\forall n \in \mathbb{N}, \quad \lim_{t \to 0^+} t^{-2} \mathrm{d}_{\mathsf{F},n}\Big(\big(\varphi(s) + tv_\varphi + \tfrac{1}{2}t^2 w_{\varphi_*,2}\big), \varphi(U \cap S)\Big) = 0.$$

  $v_\varphi := \mathrm{D}\varphi_s(v)$, and $w_{\varphi_*} := \mathrm{D}(\varphi_*)_v(w) = (w_{\varphi_*,1}, w_{\varphi_*,2})$. The set of all such w is denoted by $\mathrm{T}^2_s S$.

Spray-Invariant Sets: Definition

Let **S** be a spray on M, and $S \subset M$ a non-empty subset. A vector $v \in \mathrm{T}M$ is called a $(\mathrm{T}^2 S, \mathbf{S})$-admissible vector if

$$\tau(v) \in S \quad \text{and} \quad \mathbf{S}(v) \in \mathrm{T}^2_{\tau(v)} S.$$

The set of all such vectors is denoted by $A_{\mathbf{S},S}$ and is called the $(\mathrm{T}^2 S, \mathbf{S})$-admissible set for **S** and S.

Connection to Geodesics: For a geodesic $g \colon I \subset \mathbb{R} \to M$, and a non-empty closed subset $S \subset M$:

$$g(t) \in S \quad \text{if and only if} \quad g'(t) \in A_{\mathbf{S},S} \quad \text{for all } t \in I.$$

Definition (Spray-Invariant Set)
A subset $S \subset M$ is spray-invariant with respect to **S** if: For any geodesic $g \colon I \to M$ of **S** such that $0 \in I$, if $g(0) \in S$ and $g'(0) \in A_{\mathbf{S},S}$, then $g(t) \in S$ for all $t \in I$.

Example: Spray-Invariant Singular Space

Consider the Fréchet space $\mathcal{E} = C^\infty(\mathbb{R}, \mathbb{R})$ of smooth real-valued functions. We use a flat spray $\mathbf{S}(f, v) = (f, v, v, \mathbf{0}_\mathcal{E})$, where geodesics are affine paths:

$$\gamma(t) = f + tv.$$

Define the singular subset $\mathsf{S} = \mathsf{S}_+ \cup \mathsf{S}_-$, where:

$$\mathsf{S}_+ := \{f \in \mathcal{E} \mid \operatorname{supp}(f) \subseteq [0, \infty)\}$$
$$\mathsf{S}_- := \{f \in \mathcal{E} \mid \operatorname{supp}(f) \subseteq (-\infty, 0]\}$$

Although S_+ and S_- are smooth submanifolds, their union S is singular at the zero function (not locally homeomorphic to a linear subspace near 0).

The admissible set $A_{\mathbf{S},\mathsf{S}}$ is given by

$$A_{\mathbf{S},\mathsf{S}} = \bigcup_{f \in \mathsf{S}} \{(f, v) \in \mathrm{T}\mathcal{E} \mid v \in \mathrm{T}_f\mathsf{S}_+ \text{ or } v \in \mathrm{T}_f\mathsf{S}_-\}.$$

For any geodesic $\gamma(t) = f + tv$ with $(f, v) \in A_{\mathbf{S},\mathsf{S}}$, we find that $\gamma(t) \in \mathsf{S}$ for all $t \in \mathbb{R}$. Therefore, S is spray-invariant with respect to $\mathbf{S}$.

Projective Equivalence & Singularities

Two sprays $\mathbf{S}$ and $\overline{\mathbf{S}}$ on M are projectively equivalent if their geodesics share the same paths, differing only by an orientation-preserving reparametrization.
Formally: Any geodesic $\overline{g}$ of $\overline{\mathbf{S}}$ can be reparametrized to be a geodesic of $\mathbf{S}$, and vice versa.

Key Observation for Singular Sets: The admissible sets $A_{\mathbf{S},S}$ and $A_{\overline{\mathbf{S}},S}$ for projectively equivalent sprays $\mathbf{S}$ and $\overline{\mathbf{S}}$ generally differ when S is singular.

Example: Constructing a Projectively Equivalent Spray

Recall the singular set $\mathsf{S} = \mathsf{S}_+ \cup \mathsf{S}_-$ from the previous example in $\mathcal{E} = C^\infty(\mathbb{R}, \mathbb{R})$.

- Let $\chi(x)$ be a standard smooth bump function supported in $[-1, 1]$.
- For any $\varepsilon > 0$, define $\chi_\varepsilon(x) = \chi(2x/\varepsilon)$, supported in $[-\varepsilon/2, \varepsilon/2]$.
- Define the scalar function $\alpha(f, v)\colon \mathrm{T}\mathcal{E} \to \mathbb{R}$ by:

$$\alpha(f, v) := \int_{\mathbb{R}} \chi_\varepsilon(x) v(x)\, dx.$$

  - For a tangent vector $v(x) = \chi_\delta(x)$ with $0 < \delta \leqslant \varepsilon/2$, we have $\alpha(f, v) > 0$.
- Now, define a new spray $\tilde{\mathbf{S}}$ as: $\tilde{\mathbf{S}}(f, v) := (f, v, v, -2\alpha(f, v) \cdot v)$.
- This spray $\tilde{\mathbf{S}}$ is projectively equivalent to the flat spray $\mathbf{S}(f, v) = (f, v, v, \mathbf{0}_{\mathcal{E}})$ used in the previous example.
  - This yields a projectively equivalent spray since it modifies the second derivative by a multiple of the adjacent tangent vector.

S is not spray-invariant under $\tilde{\mathbf{S}}$.

Theorem: Totally Geodesic Submanifolds

Theorem (Characterization of Totally Geodesic Submanifolds)

Let $\mathbf{S}$ *be a spray on a manifold* M, *and let* S *be a* C^3*-submanifold of* M. *Then* S *is totally geodesic if and only if* $A_{\mathbf{S},S} = \mathrm{T}S$.

Corollary

Let M *be a manifold such that, given any two distinct points in* M, *there is a unique geodesic passing through them. Let* $S \subset \mathrm{M}$ *be a closed* C^3*-submanifold with the property that, locally, given any two distinct points in* S, *the unique geodesic segment in* M *connecting them lies entirely in* S. *Then* S *is a totally geodesic submanifold of* M.

Example: Spray-Invariant but Not Totally Geodesic

Let $\mathcal{M} = C^\infty(\mathbb{R}, \mathbb{R}^2)$ be the Fréchet space of smooth $\mathbb{R}^2$-valued functions. Equipped with the flat spray $\mathbf{S}(f, v) = (f, v, v, (0, 0))$, where $(0, 0)$ is the zero function in $\mathcal{M}$. Consider the subset $S \subseteq \mathcal{M}$ defined by

$$S := \{f \in \mathcal{M} \mid f = (h, h^2) \text{ for some } h \in E = C^\infty(\mathbb{R}, \mathbb{R})\}.$$

The admissible set $A_{\mathbf{S},S}$ for this spray and subset is

$$A_{\mathbf{S},S} = \{(f, 0) \in \mathrm{T}\mathcal{M} \mid f = (h, h^2),\ h \in E\}.$$

The tangent bundle $\mathrm{T}S$ is

$$\mathrm{T}S = \{(f, v) \in \mathrm{T}\mathcal{M} \mid f = (h, h^2),\ v = (u, 2hu)) \text{ for some } u \in E\}.$$

S is spray-invariant. However, as $\mathrm{T}S \neq A_{\mathbf{S},S}$, this means S is NOT totally geodesic.

Example: A Totally Geodesic Submanifold

Let $\mathcal{M} = C^\infty(\mathbb{R}^n, \mathbb{R})$ be the Fréchet space of smooth real-valued functions on $\mathbb{R}^n$. We use the flat spray $\mathbf{S}(f, v) = (f, v, v, \mathbf{0}_\mathcal{M})$, where geodesics are affine paths:

$$\gamma(t) = f + tv.$$

Define the subset $S \subset \mathcal{M}$ as the set of constant functions on $\mathbb{R}^n$:

$$S := \{f \in C^\infty(\mathbb{R}^n, \mathbb{R}) \mid \exists\, c \in \mathbb{R} \text{ such that } f(x) = c,\ \forall x \in \mathbb{R}^n\}.$$

The set S can be identified with $\mathbb{R}$. It's a closed linear subspace of $\mathcal{M}$, making it a closed C^∞-submanifold.

Since all conditions of the corollary are satisfied (unique geodesics, and locally, geodesics between points in S stay in S), S is a totally geodesic submanifold.

Automorphisms of Sprays

A C^k-automorphism ϕ of M is an automorphism of the spray $\mathbf{S}$ if

$$\phi_{**} \circ \mathbf{S} \circ \phi_*^{-1} = \mathbf{S}.$$

These automorphisms form a group under composition, denoted by $\mathrm{Aut}(\mathrm{M}, \mathbf{S})$. They preserve the spray's structure.

Theorem (Automorphisms Preserve Spray-Invariance)

Let $S \subset \mathrm{M}$ be a non-empty closed subset that is spray-invariant with respect to $\mathbf{S}$, and let $\phi \in \mathrm{Aut}(\mathrm{M}, \mathbf{S})$. Then $\phi(S)$ is also spray-invariant with respect to $\mathbf{S}$.

Example: Translation Automorphism

Recall the singular spray-invariant set $\mathrm{S} = \mathrm{S}_+ \cup \mathrm{S}_-$ from previous examples in the Fréchet space $\mathcal{E} = C^\infty(\mathbb{R}, \mathbb{R})$, with the flat spray $\mathbf{S}$.

- For a fixed $a \in \mathbb{R}$, $a \neq 0$, consider the translation map:

$$\phi_a \colon \mathcal{E} \to \mathcal{E}, \quad \phi_a(f)(x) = f(x - a).$$

- This translation map ϕ_a is an automorphism of the flat spray $\mathbf{S}$.
  - Geodesics of the flat spray are affine paths: $\gamma(t) = f + tv$.
  - Applying ϕ_a to a geodesic just translates the function along the x-axis: $\phi_a(\gamma(t))(x) = \gamma(t)(x - a) = f(x - a) + tv(x - a)$.
  - This transformed curve is still an affine path in $\mathcal{E}$ and thus a geodesic of the flat spray.
- Since S is spray-invariant and $\phi_a \in \mathrm{Aut}(\mathcal{E}, \mathbf{S})$, by the theorem:
  - The translated set $\phi_a(\mathrm{S})$ is also spray-invariant.

Invariance of $A_{\mathbf{S},S}$

Key Question: If S is spray-invariant, does the spray **S** (regarded as a first-order vector field on TM) remain second-order adjacent tangent to its own admissible set $A_{\mathbf{S},S}$?

Why this matters: This reformulation reduces the problem from analyzing second-order dynamics on M to studying first-order dynamics on TM, which can be more tractable.

Potential Tool: The Nagumo-Brezis Theorem provides a criterion for determining the invariance of sets under the flow of a vector field.

- The theorem's classical formulation applies primarily to Banach manifolds.
- It does not generalize straightforwardly to arbitrary Fréchet manifolds, posing a significant hurdle for our infinite-dimensional setting. This theorem holds true for the category of MC^k-Fréchet manifolds under nuclearity assumptions.

Bounded Differentiability: Foundations

To define MC^k-differentiability (defined by Olaf Müller, 2006) in Fréchet spaces E and F:

- We equip E and F with a specific translation-invariant metric:

$$\mathrm{m_F}(x, y) = \sup_{n \in \mathbb{N}} \frac{1}{2^n} \frac{\|x - y\|_{F,n}}{1 + \|x - y\|_{F,n}}.$$

- We consider the space $\mathrm{L}(\mathrm{E}, \mathrm{F})$ of linear mappings $L \colon \mathrm{E} \to \mathrm{F}$ that are (globally) Lipschitz continuous with respect to these metrics.
  - The Lipschitz constant of a linear map L is

$$\mathrm{Lip}(L) := \sup_{x \in \mathrm{E} \setminus \{\mathbf{0}_\mathrm{E}\}} \frac{\mathrm{m_F}(L(x), \mathbf{0}_\mathrm{F})}{\mathrm{m_E}(x, \mathbf{0}_\mathrm{E})} < \infty.$$

- We equip $\mathrm{L}(\mathrm{E}, \mathrm{F})$ with its own translation-invariant metric:

$$(L, H) \mapsto \mathrm{Lip}(L - H).$$

Bounded Differentiability: Definition

- Let $\varphi\colon U \subseteq \mathsf{E} \to \mathsf{F}$ be a C^1-mapping (in the sense of Michal-Bastiani).
- φ is called bounded differentiable (or MC^1-differentiable) if
  1. The directional derivative $D\varphi(x)$ (which is a linear map from E to F) belongs to the space of Lipschitz linear maps $\mathsf{L}(\mathsf{E}, \mathsf{F})$ for all $x \in U$.
  2. The induced mapping $D\varphi\colon U \to \mathsf{L}(\mathsf{E}, \mathsf{F})$, which sends $x \mapsto D\varphi(x)$, is continuous with respect to the defined metrics on U and $\mathsf{L}(\mathsf{E}, \mathsf{F})$.

In essence:

- This differentiability requires not just existence of directional derivatives, but also that these derivatives are *Lipschitz continuous* linear maps, and that the derivative mapping itself is continuous in a specific metric.
- This is a stronger form of differentiability.

Nuclear Spaces

Let $(B_1, |\cdot|_1)$ and $(B_2, |\cdot|_2)$ be Banach spaces. A linear operator $T : B_1 \to B_2$ is called nuclear if it can be written in the form:

$$T(x) = \sum_{j=1}^{\infty} \lambda_j \langle x, x_j \rangle y_j$$

- $\langle \cdot, \cdot \rangle$ is the duality pairing between B_1 and its dual B_1'.
- $x_j \in B_1'$ with $| x_j |_1' \leqslant 1$.
- $y_j \in B_2$ with $| y_j |_2 \leqslant 1$.
- λ_j are complex numbers such that $\sum_j | \lambda_j | < \infty$.

A nuclear space is a locally convex topological vector space such that every continuous linear map to a Banach space is nuclear.

Flow-Invariance & The Nagumo-Brezis Theorem

- Let $A \subset \mathrm{M}$ and V be an MC^1-vector field on M.
- The set A is called flow-invariant with respect to V if:
  - For any integral curve $I(t)$ of V, if $I(0) \in A$, then $I(t) \in A$ for all $t \geqslant 0$ within the curve's domain.

Theorem (Nagumo-Brezis Theorem, K.E., 2024)

Let M *be a nuclear* MC^k*-Fréchet manifold, and let* $\mathrm{V} \colon \mathrm{M} \to \mathrm{TM}$ *be an* MC^1*-vector field. Let* $A \subset \mathrm{M}$ *be closed. Then,* A *is flow-invariant with respect to* V *if and only if for each* $x \in \mathrm{M}$*, there exists a chart* (U, ϕ) *around* x*, such that:*

$$\lim_{t \to 0^+} t^{-1} \mathrm{m}_{\mathsf{F}} \left(\phi(x) + t \mathrm{D}\phi(x)(\mathrm{V}(x)), \phi(U \cap A) \right) = 0.$$

This condition essentially means $\mathrm{V}(x)$ *is an adjacent tangent vector to* $\phi(U \cap A)$ *at* $\phi(x)$*.*

Linking Spray-Invariance to Adjacent Tangency

Recall: The question was: If S is spray-invariant, does **S** (as a vector field on TM) remain second-order adjacent tangent to $A_{\mathbf{S},S}$?

Theorem

Let M *be a nuclear* MC^k*-Fréchet manifold, and let* $S \subset \mathrm{M}$ *be a subset such that* $A_{\mathbf{S},S}$ *is non-empty and closed. Then, the following are equivalent:*

1. S *is spray-invariant with respect to* **S**.
2. **S** *is adjacent tangent to* $A_{\mathbf{S},S}$ *when regarded as a vector field on* TM.

Theorem

Let B *be a* C^k*-Banach manifold,* $k \geqslant 4$*, and* $S \subset \mathrm{B}$ *a subset such that* $A_{\mathbf{S},S}$ *is non-empty and closed. Then,* S *is spray-invariant if and only if* **S** *is adjacent tangent to* $A_{\mathbf{S},S}$ *when regarded as a vector field on* TB.

Example: Spray-Invariant Set of Non-Negative Functions

Consider the Banach manifold $\mathcal{M} = C^k(S^1, \mathbb{R})$ of k-times differentiable functions on the circle S^1. We equip it with the flat spray $\mathbf{S}$, whose geodesics are affine paths:

$$\gamma(t) = f + tv, \quad f \in \mathcal{M},\ v \in \mathrm{T}_f\mathcal{M}.$$

Let $S \subset \mathcal{M}$ be the closed subset of non-negative functions:

$$S := \left\{ f \in \mathcal{M} \mid f(\theta) \geqslant 0,\ \ \forall \theta \in S^1 \right\}.$$

When viewed as a vector field on $\mathrm{T}\mathcal{M}$, the spray $\mathbf{S}$ satisfies $\mathbf{S}(v) = 0$ for all $v \in A_{\mathbf{S},S}$.

- Since the zero vector lies in every adjacent cone, this means $\mathbf{S}$ is adjacent tangent to $A_{\mathbf{S},S}$.

As $\mathbf{S}$ is adjacent tangent to $A_{\mathbf{S},S}$ (and $A_{\mathbf{S},S}$ is closed), this implies that S is spray-invariant. The set S is also a convex cone with vertex at the zero function.

Theorem: Transversality and Spray-Invariance

Theorem

Let M *be a nuclear* MC^k*-Fréchet manifold. Let* S *be a closed* MC^3*-submanifold of* M *(as introduced in previous examples).*

If the spray $\mathbf{S}$ *restricted to* $\mathrm{T}S$ *is transverse to* $\mathrm{T}(\mathrm{T}S)$*, i.e.,*

$$\mathbf{S}\big|_{\mathrm{T}S} \pitchfork \mathrm{T}(\mathrm{T}S),$$

then S *is spray-invariant with respect to* $\mathbf{S}$ *if and only if*

$$\forall v \in \mathbf{S}(\mathrm{T}(\mathrm{T}S)), \quad \mathrm{D}\mathbf{S}(v)(\mathbf{S}(v)) \in \mathrm{T}_{\mathbf{S}(v)}(\mathrm{T}(\mathrm{T}S)). \tag{1}$$

Geodesic Flow Invariance Theorem

- Assume B is a Banach manifold of class C^k ($k \geqslant 4$), and $\mathbf{S}$ is a spray on B of class C^2.
- The geodesic flow is the mapping:

$$\Phi_t \colon \mathrm{TB} \to \mathrm{TB}$$

that satisfies $\Phi_t(v) = g_v'(t)$, where $g_v \colon I \to \mathrm{B}$ is the unique geodesic with initial tangent $v \in \mathrm{TB}$.

Theorem

A closed subset $S \subset \mathrm{B}$ is spray-invariant if and only if its admissible set $A_{\mathbf{S},S}$ is invariant under the geodesic flow Φ_t.

Example: Constant Maps to a Great Circle

Consider the Hilbert manifold $\mathcal{M} = L^2(S^1, S^2)$, the space of square-integrable maps from the circle S^1 into the 2-sphere S^2. The tangent space at $f \in \mathcal{M}$ is:

$$\mathrm{T}_f\mathcal{M} \cong L^2(S^1, \mathrm{T}_{f(\theta)}S^2).$$

$\mathcal{M}$ carries the natural L^2-Riemannian metric:

$$\langle v, w \rangle_f = \int_{S^1} \langle v(\theta), w(\theta) \rangle_{g_{S^2}(f(\theta))} \, d\theta.$$

Let $\mathbf{S}$ be the canonical spray associated with this metric.

Let $C \subset S^2$ be a great circle (a totally geodesic submanifold of S^2). Define the closed subset $S := \left\{ f \in \mathcal{M} \,\middle|\, \exists p \in C \text{ such that } f(\theta) = p \text{ for almost all } \theta \in S^1 \right\}$. Its admissible set $A_{\mathbf{S},S}$ is invariant under the geodesic flow Φ_t. Therefore, by the Theorem, S is spray-invariant. However, S is NOT totally geodesic.

Example: Space of Finite Sequences (ℓ^2) - Setup

Let $H = \ell^2$, the separable Hilbert space of square-summable real sequences, with the standard inner product:

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i.$$

Let $\{e_n\}_{n\in\mathbb{N}}$ be its standard orthonormal basis. Define the subset $\mathcal{S} := \{x \in H \mid \text{only finitely many coordinates of } x \text{ are nonzero}\}$.

- This is the space of finite sequences.
- It can be expressed as a countable union of finite-dimensional linear subspaces:

$$\mathcal{S} = \bigcup_{k=1}^{\infty} H_k, \quad \text{where } H_k := \operatorname{span}(e_1, \ldots, e_k).$$

Consider the flat spray of ℓ^2. Let $x \in \mathcal{S}$ and $v \in \mathrm{T}_x\mathcal{S}$. Then there exists a k such that both $x, v \in H_k$. The geodesic starting at x with tangent v is given by $\gamma(t) = x + tv$. Since H_k is a linear subspace, $\gamma(t) \in H_k \subset \mathcal{S}$ for all $t \in \mathbb{R}$. $\mathcal{S}$ is spray-invariant.

Example: Stratification

- We consider a stratification of $\mathcal{S}$ into strata S_i:

$$S_i = H_i \backslash H_{i-1},$$

- We now verify the frontier axiom for this stratification, where the closure is taken with respect to the topology induced from ℓ^2.
- The closure of a stratum S_i in $\mathcal{S}$ is simply $\overline{S_i} = H_i$.
    - **Case 1:** $i < j$
      $\overline{S_i} = H_i$. Since $H_i \subset H_j$, but H_i contains vectors with at most i nonzero components, while S_j contains vectors with exactly $j > i$ nonzero components, it follows that $H_i \cap S_j = \emptyset$. Thus, $\overline{S_i} \cap S_j = \emptyset$.
    - **Case 2:** $i = j$
      Trivially, $\overline{S_i} = H_i$, and $\overline{S_i} \cap S_i = S_i \neq \emptyset$. Furthermore, $S_i \subset \overline{S_i}$ by definition.
    - **Case 3:** $i > j$
      We have $H_j \subset H_i$, and $S_j = H_j \backslash H_{j-1} \subset H_i$. Hence, $\overline{S_i} \cap S_j = S_j \neq \emptyset$, and $S_j \subset \overline{S_i}$.

G-Invariant Sprays

Let G be a smooth Lie group acting smoothly on a smooth Banach manifold B. A spray $\mathbf{S}$ on B is called *G-invariant* if, for every $g \in G$, the action of g on B lifts to a smooth transformation $\mathrm{T_g} : \mathrm{TB} \to \mathrm{TB}$ such that $\mathbf{S}$ is preserved under this lifted action. More precisely, for all $g \in G$, the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{TTB} & \xrightarrow{\mathrm{T(T_g)}} & \mathrm{TTB} \\
\Big\downarrow \mathbf{S} & & \Big\downarrow \mathbf{S} \\
\mathrm{TB} & \xrightarrow{\mathrm{T_g}} & \mathrm{TB}
\end{array}
$$

his condition means that for any $v \in \mathrm{TB}$, we have

$$\mathrm{T(T_g)}(\mathbf{S}(v)) = \mathbf{S}(\mathrm{T_g}(v)).$$

For a point $x \in$ B, the isotropy group (or stabilizer) of x, denoted by G_x, is the subgroup of G consisting of all elements $g \in G$ that leave x unchanged under the group action:

$$G_x = \{g \in G \mid g \cdot x = x\}.$$

Definition of a Slice: G-Action

A slice at $x \in \mathrm{B}$ is a submanifold $V \subset \mathrm{B}$ containing x such that

1. ***H*-Invariance:** $h \cdot v \in V$ for all $h \in H$ and $v \in V$, where $H = G_x$.
2. **Local Triviality:** There exists a G-equivariant diffeomorphism

$$\Phi \colon G \times_H V \to U$$

onto a G-equivariant open neighborhood $U \subset \mathrm{B}$ of the orbit $G \cdot x$, such that $\Phi([g, v]) = g \cdot v$ and $\Phi([e, x]) = x$, where e is the identity in G.

3. **Transversality:**
   3.1 $\mathrm{T}_x V \cap \mathrm{T}_x(G \cdot x) = \{0\}$.
   3.2 $\mathrm{T}_x V$ is a closed subspace of $\mathrm{T}_x \mathrm{B}$ such that $\mathrm{T}_x \mathrm{B} = \mathrm{T}_x(G \cdot x) \oplus \mathrm{T}_x V$.
   3.3 The map $\alpha \colon G \times V \to \mathrm{B}$, given by $\alpha(g, v) = g \cdot v$, has a derivative at (e, x),

$$\mathrm{T}_{(e,x)}\alpha \colon \mathrm{T}_e G \times \mathrm{T}_x V \to \mathrm{T}_x \mathrm{B},$$

which is surjective, with kernel complemented in $\mathrm{T}_e G \times \mathrm{T}_x V$.

Theorem: Orbit Type Stratification & Spray-Invariance

Theorem

Let G be a finite-dimensional smooth Lie group acting smoothly on a smooth Banach manifold B*. Assume that a smooth spray* $\mathbf{S}$ *on* B *is* *G-invariant. And assume that for every $x \in \mathsf{B}$, there exists a G-equivariant neighborhood U of x and a G-equivariant diffeomorphism*

$$\Phi \colon G \times_H V \to U$$

where V is a slice at x and $H = G_x$ is the isotropy subgroup. Then the orbit type decomposition of B*, given by*

$$\mathsf{B} = \bigcup_{[H]} \mathsf{B}_{(H)}, \quad \textit{where } \mathsf{B}_{(H)} = \{x \in \mathsf{B} \colon G_x \cong H\},$$

defines a stratification of B *such that each stratum* $\mathsf{B}_{(H)}$ *is spray-invariant.*

Nuance: Strata vs. Individual Orbits

An important distinction arises from Theorem 10:

- The theorem guarantees that geodesics starting in an orbit type stratum $\mathrm{B}_{(H)}$ remain in that stratum.
    - This means if you start at a point x with isotropy group H, the geodesic through x will always stay within the set of points whose isotropy group is isomorphic to H.
- **However, this does not imply that geodesics remain in the same individual orbit.**
    - A geodesic starting at x might move to other points y within the same stratum $\mathrm{B}_{(H)}$ such that y is *not* in the same orbit as x (i.e., $y \notin G \cdot x$), but y still has an isotropy group isomorphic to H.
- Spray-invariance applies at the level of strata (sets of points with isomorphic isotropy groups), not necessarily at the finer level of individual orbits.

Thank You!