Constant mean curvature surfaces with harmonic Gauss maps in three-dimensional Lie groups

(Based on a joint work with Iryna Savchuk)

Eugene Petrov

V.N. Karazin Kharkiv National University, Kharkiv, Ukraine

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Let M be an (immersed smooth) hypersurface in an $(n+1)$-dimensional Lie group G with a left-invariant Riemannian metric. Consider a natural generalization of the Gauss map of a hypersurface in $\mathbb{E}^{n+1}$:

Definition

The *left-invariant Gauss map* $\Phi \colon M \to \mathbb{R}P^n$ of M is defined by

$$\Phi(p) = d_p L_{p^{-1}}(N_p M),$$

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Definition

The left-invariant Gauss map $\Phi\colon M \to \mathbb{S}^n$ of an oriented M is defined by

$$\Phi(p) = d_p L_{p^{-1}}(\eta_p),$$

where η is the unit normal vector field of M.

Here elements of $\mathbb{R}P^n$ and $\mathbb{S}^n$ are identified with 1-dimensional subspaces and unit vectors respectively in the Lie algebra $\mathfrak{g}$ of G.

Let (M, g) and (N, h) be Riemannian manifolds, M be compact.

Definition

A map $\Phi\colon M \to N$ is called *harmonic* if it is a stationary point of the *energy functional*

$$\Phi \mapsto E(\Phi) = \int\limits_M e(\Phi) dV_g,$$

where $e(\Phi) = \frac{1}{2} \operatorname{Tr}_g(\Phi^* h)$ is the *energy density function*.

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The corresponding Euler-Lagrange equation has the form

$$\tau(\Phi) = \operatorname{Tr}_g B_\Phi = 0,$$

where $B_\Phi \in \Gamma(TM^* \otimes TM^* \otimes \Phi^{-1} TN)$ is the *second fundamental form* of Φ, defined in the obvious notation by

$$B_\Phi(X, Y) = (\nabla_h)_X d\Phi(Y) - d\Phi((\nabla_g)_X Y).$$

The field $\tau(\Phi) = \mathrm{Tr}_g\, B_\Phi \in \Gamma(\Phi^{-1}TN)$ is called the *tension field* of Φ. Using it we can extend the definition to the case of non-compact M:

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In particular,

Proposition

- $\Phi\colon M \to \mathbb{E}^n$ is harmonic if and only if $\Delta_g \Phi^a = 0$ for all a.
- (T. Takahashi, 1966) $\Phi\colon M \to \mathbb{S}^n \hookrightarrow \mathbb{E}^{n+1}$ is harmonic if and only if $\Delta_g \Phi^a = -2e(\Phi)\Phi^a$ for all a.

Here Δ_g is the Riemannian Laplacian of (M, g).

Let's endow $\mathbb{R}P^n$ (or $\mathbb{S}^3$) with its standard Riemannian metric of constant curvature.

Theorem (E.A. Ruh, J. Vilms, 1970)

The Gauss map Φ of a connected hypersurface M in $\mathbb{E}^{n+1}$ is harmonic if and only M is of constant mean curvature (CMC).

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Theorem (N. do Espírito-Santo, S. Fornari, K. Frensel, J. Ripoll, 2011)

The (left-invariant) Gauss map Φ of a connected hypersurface M in a Lie group G with a biinvariant metric is harmonic if and only M is CMC.

In particular, it is true for groups $\mathbb{E}^{n+1}$, $\mathrm{SO}(3)$ (and its universal covering $\mathbb{S}^3 \cong \mathrm{SU}(2)$) with metrics of constant curvature.
In fact, the Ruh – Vilms theorem is true for submanifolds of an arbitrary codimension with $\nabla H = 0$, but its higher-codimension generalizations to the Lie groups do not take place even for biinvariant metrics and totally geodesic submanifolds.

The next example shows that the generalization of Ruh – Vilms theorem is not true for general left-invariant metrics and left-invariant Gauss maps (in the following, we will consider such maps by default).

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The $(2m+1)$-dimensional *Heisenberg Lie group* Nil^{2m+1} can be identified with $\mathbb{R}^{2m+1}$ in such a way that left-invariant fields

$$X_i = \frac{\partial}{\partial x^i} - \frac{y^i}{2}\frac{\partial}{\partial z},\ i = \overline{1,m},\ \ Y_i = \frac{\partial}{\partial y^i} + \frac{x^i}{2}\frac{\partial}{\partial z},\ i = \overline{1,m},\ \ Z = \frac{\partial}{\partial z}.$$

form a basis of its Lie algebra with non-zero brackets $[X_i, Y_i] = Z$, $i = \overline{1,m}$. Consider the left-invariant metric on Nil^{2m+1} such that this frame is orthonormal.

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Theorem (E. P., 2011)

If M is a CMC hypersurface in Nil^{2m+1} with the harmonic Gauss map Φ then M is *vertical*, i.e., the field Z is tangent to it. If it is complete then $M = M_1 \times \mathbb{R}$ is a cylinder over a CMC hypersurface M_1 in $\mathbb{E}^{2n}$.

In particular, for $m = 1$ complete connected CMC surfaces in Nil^3 with harmonic Gauss maps are vertical Euclidean planes and vertical Euclidean round cylinders.

To prove the previous theorem we obtained a general criterion of harmonicity for Φ. For surfaces it takes the following form:

Proposition

Let M be a surface in an 3-dimensional Lie group G with a left-invariant Riemannian metric $\langle\cdot,\cdot\rangle$, its Riemannian connection ∇ and its Ricci tensor Ric. Let $\{Y_1, Y_2\}$ be an orthonormal frame of T_pM at some $p \in M$, $\{b_{ij}\}$ be the coefficients of the second fundamental form of M at p with respect to this basis, H be its mean curvature, and Y_3 be a unit normal vector to M at p. Denote by $\{Y_1, Y_2, Y_3\}$ also the continuations of these vectors by left-invariant fields on G. Then the Gauss map Φ of M is harmonic at p (i.e., its tension field vanishes at p) if and only if for $j = 1, 2$

$$\mathrm{Ric}(Y_3, Y_j) + \sum_{1\leqslant i\leqslant 2} \left\langle \nabla_{\nabla_{Y_i} Y_i} Y_3 + \nabla_{Y_i} \nabla_{Y_i} Y_3, Y_j \right\rangle -$$

$$- Y_j(2H) - 2H \left\langle \nabla_{Y_3} Y_3, Y_j \right\rangle + 2 \sum_{1\leqslant i,k\leqslant n} b_{ik} \left\langle \nabla_{Y_i} Y_k, Y_j \right\rangle = 0.$$

Let us use a well-known description by J. Milnor (1976) of left invariant metrics on a three-dimensional connected Lie group G. If G is unimodular, i.e., $\mathrm{Tr}\,\mathrm{ad}_x = 0$ for each $x \in \mathfrak{g}$, then there exists an orthonormal frame $\{e_1, e_2, e_3\}$ of $\mathfrak{g}$ such that

$$[e_2, e_3] = \lambda_1 e_1,\ [e_3, e_1] = \lambda_2 e_2,\ [e_1, e_2] = \lambda_3 e_3$$

for some $\{\lambda_i\} \subset \mathbb{R}$. Then the formulae from the previous proposition can be rewritten in the terms of these coefficients.

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The equality $\lambda_1 = \lambda_2 = \lambda_3$ means that $\mathrm{ad}_x = [x, \cdot]$ is skew-adjoint for each $x \in \mathfrak{g}$, i.e., the metric is biinvariant. These are constant curvature metrics on $\mathbb{E}^3$ (for $\lambda_i = 0$), $\mathrm{SO}(3)$ and its universal covering $\mathbb{S}^3 \cong \mathrm{SU}(2)$ (for $\lambda_i \neq 0$). For this case the harmonicity criterion takes the form

$$Y_1(2H) = Y_2(2H) = 0,$$

so indeed Φ is harmonic if and only if M is CMC.

Let only two of $\{\lambda_i\}$ be equal, say (without loss of generality) $\lambda_1 = \lambda_1 = \lambda \neq \mu = \lambda_3$. This means that ad_x is skew-adjoint only for x proportional to e_3, so the metric is right invariant with respect to a one-dimensional subgroup $H = \exp(\mathbb{R}e_3)$, but not biinvariant.

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In this case Φ is harmonic if and only if

$$Y_1(2H) + (\lambda - \mu) \sin 2\alpha \,(H + b_{22}) = 0,$$

$$Y_2(2H) - \frac{1}{2}(\lambda - \mu) \sin 2\alpha \left((\lambda - \mu) \sin^2 \alpha + 2b_{12}\right) = 0,$$

where α is the angle between Y_3 and e_3.

This allows us to prove the following:

Theorem

Let a left invariant metric on a connected three-dimensional unimodular Lie group G be right invariant with respect to a one-dimensional subgroup $H \subset G$, but not biinvariant, and let M be a connected surface in G. Then from any two of the following claims the third follows:

1. M is CMC;
2. the Gauss map Φ of M is harmonic;
3. M is either everywhere orthogonal to the one-dimensional foliation generated by H (is *horizontal*) or is composed of leaves of this foliation (is *vertical*).

Indeed, if the claim 3 holds then $\sin 2\alpha = 0$, so claims 1 and 2 are equivalent. If for a CMC surface M with harmonic Φ we assume that the claim 3 is not true, we can express $\{b_{ij}\}$ from the equalities above, and then the Codazzi equations lead to a contradiction.

In particular, $\lambda = 0$, $\mu \neq 0$ corresponds to the 3-dimensional Heisenberg group Nil^3. As the orthogonal distribution to e_3 is non-integrable, there are only vertical CMC surfaces with harmonic Φ, so a previous theorem for the 3-dimensional case follows.

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For $\lambda \neq 0$, $\mu = 0$ we have the group $\mathrm{E}(2)$ of orientation-preserving Euclidean plane isometries. It is solvable, so its universal cover $\widetilde{\mathrm{E}(2)}$ can be identified with $\mathbb{R}^3$, and the metric is Euclidean. The integral trajectories of e_3 are vertical straight lines.

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Corollary

Any connected complete CMC surface in $\widetilde{\mathrm{E}(2)}$ with the harmonic Gauss map is a vertical Euclidean round cylinder or a vertical or horizontal Euclidean plane.

Non-zero λ and μ of different signs correspond to the group $\mathrm{SL}(2,\mathbb{R})$. In particular, for $\lambda = 1$, $\mu = -1$ we have the standard metric (a Thurston geometry) on $\mathrm{SL}(2,\mathbb{R})$ and its universal covering $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. The latter is also the universal covering of $T_1\mathbb{H}^2$ with the Sasaki metric, i.e., the half-space $\{(x,y,z) \in \mathbb{R}^3 \mid y > 0\}$, and

$$e_1 = y\cos z\frac{\partial}{\partial x} + y\sin z\frac{\partial}{\partial y} - \cos z\frac{\partial}{\partial z},$$

$$e_2 = -y\sin z\frac{\partial}{\partial x} + y\cos z\frac{\partial}{\partial y} + \sin z\frac{\partial}{\partial z},\ e_3 = \frac{\partial}{\partial z},$$

so the integral trajectories of e_3 are vertical straight lines. Their orthogonal distribution is again non-integrable.

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Corollary

Any connected complete CMC surface in $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ with the harmonic Gauss map is a vertical Euclidean round cylinder or a vertical Euclidean plane.

Finally, non-zero pairs $\lambda \neq \mu$ of the same sign give non-constant curvature left-invariant metrics on $\mathbb{S}^3$.

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The case of pairwise different $\{\lambda_i\}$ is considerably harder. For example, $\lambda_1 = -\lambda_2 = 1$, $\lambda_3 = 0$ gives the standard metric (again a Thurston geometry) on the solvable group Sol^3. If we identify it with $\mathbb{R}^3$ then the frame takes the form

$$e_1 = \frac{1}{\sqrt{2}}\left(e^{-z}\frac{\partial}{\partial x} + e^{z}\frac{\partial}{\partial y}\right),\ e_2 = \frac{1}{\sqrt{2}}\left(e^{-z}\frac{\partial}{\partial x} - e^{z}\frac{\partial}{\partial y}\right),\ e_3 = \frac{\partial}{\partial z}.$$

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Using some examples of minimal surfaces found by L. Masaltsev (2006), we can get from our harmonicity criterion that, in particular,

- the minimal surface $z = 0$ has harmonic Gauss map;
- the totally geodesic surfaces $x = 0$ and $y = 0$ have non-harmonic Gauss maps.

If G is not unimodular then there exist an orthonormal frame $\{e_1, e_2, e_3\}$ on $\mathfrak{g}$ such that

$$[e_2, e_3] = (1 - \lambda)(\mu e_1 - e_2),\ [e_3, e_1] = (1 + \lambda)(e_1 + \mu e_2),\ [e_1, e_2] = 0$$

for some $\lambda, \mu \in \mathbb{R}$.

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for some $\lambda, \mu \in \mathbb{R}$.
In particular, for $\lambda = \mu = 0$ the sectional curvature of this metric is -1, so we get the Lie group structure on the hyperbolic space $\mathbb{H}^3$. For the half-space $(z > 0)$ model with the usual metric $\frac{1}{z^2}(dx^2 + dy^2 + dz^2)$ we then have

$$e_1 = z\frac{\partial}{\partial x},\ e_2 = z\frac{\partial}{\partial y},\ e_3 = z\frac{\partial}{\partial z}.$$

In this case Φ is harmonic if and only if

$$Y_1(H) + \sin\alpha\,(H - b_{11}) = 0,$$

$$Y_2(H) + \sin\alpha\, b_{12} = 0,$$

where α is the angle between Y_3 and e_3.

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where α is the angle between Y_3 and e_3.
Then similarly to the previous theorem we can prove

Theorem

A complete connected surface in the hyperbolic space $\mathbb{H}^3$ is CMC with the harmonic Gauss map if and only if it is a horosphere $z = z_0$ parallel to the sphere at infinity.

Thank you!

Selected references

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