A Banach space characterization of (sequentially) Ascoli spaces

Saak Gabriyelyan

Ben Gurion University of the Negev

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Ascoli Spaces. Motivation and Definition.

Let X be a Tychonoff space. Denote by $C_k(X)$ the space $C(X)$ of all real or complex valued continuous functions on X endowed with the compact-open topology. Recall that

Definition 1: X is called a *k-space* if for each non-closed subset $A \subseteq X$ there is a compact subset $K \subseteq X$ such that $A \cap K$ is not closed in K.

A map S from X to a Tychonoff space Y is called *k-continuous* if each restriction $f{\restriction}_K$ of f to any compact subset K of X is continuous. We recall that

Definition 2: X is called a *$k_{\mathbb{R}}$-space* if every k-continuous function $f : X \to \mathbb{R}$ is continuous.

It is well known that X is a $k_{\mathbb{R}}$-space if and only if $C_k(X)$ is complete. Each k-space is a $k_{\mathbb{R}}$-space, but there are $k_{\mathbb{R}}$-spaces which are not k-spaces (for example, $\mathbb{R}^{\aleph_1}$ is a $k_{\mathbb{R}}$-space which is not a k-space).

One of the basic theorems in Analysis is the Ascoli theorem (see Theorem 3.4.20 in [2]) which states that if X is a k-space, then every compact subset of $C_k(X)$ is evenly continuous, that is the map $X \times \mathcal{K} \ni (x, f) \mapsto f(x)$ is continuous. In [13], Noble proved that every $k_{\mathbb{R}}$-space satisfies the conclusion of the Ascoli theorem.

Ascoli Spaces. Motivation and Definition.

So it is natural to consider the class of Tychonoff spaces which satisfy the conclusion of Ascoli's theorem. Following Banakh and Gabriyelyan [1]:

Definition 3: A Tychonoff space X is called an *Ascoli space* if every compact subset $\mathcal{K}$ of $C_k(X)$ is evenly continuous.

In other words, X is Ascoli if and only if the compact-open topology of $C_k(X)$ is Ascoli in the sense of [12, p.45]. There are Ascoli spaces which are not $k_{\mathbb{R}}$-spaces. The first such example is given in [1]. Another example is $C_p(\omega_1)$.

One can easily show that X is Ascoli if and only if every compact subset of $C_k(X)$ is equicontinuous. Recall that a subset H of $C(X)$ is *equicontinuous* if for every $x \in X$ and each $\varepsilon > 0$ there is an open neighborhood U of x such that $|f(x') - f(x)| < \varepsilon$ for all $x' \in U$ and $f \in H$.

Being motivated by the classical notion of c_0-barrelled locally convex spaces we defined in [6]

Definition 4: A Tychonoff space X to be *sequentially Ascoli* if every convergent sequence in $C_k(X)$ is equicontinuous.

Clearly, every Ascoli space is sequentially Ascoli, but the converse is not true in general (every non-discrete P-space is sequentially Ascoli but not Ascoli, see [6]).

Ascoli Spaces. Motivation.

Ascoli and sequentially Ascoli spaces in various classes of topological, function and locally convex spaces are thoroughly studied in [1, 3, 4, 5, 7, 8, 9].

A map T from a Tychonoff space X to a locally convex space E is called *bounded* if $T(X)$ is a bounded subset of E. The space E endowed with the weak topology is denoted by E_w.

Our motivation is the following characterization of sequentially Ascoli spaces proved in [10].

Theorem 5: *A Tychonoff space X is sequentially Ascoli if and only if every k-continuous bounded map $T : X \to c_0$ is continuous whenever it is continuous as a map from X into $(c_0)_w$.*

To understand how to use Theorem 5, we observe first that c_0 is a subspace of the the dual Banach space ℓ_∞ of ℓ_1. Therefore a map $T : X \to c_0$ can be considered as a map from X into the Banach dual ℓ_∞ of ℓ_1. Consequently, the space $(c_0)_w$ is a subspace of ℓ_∞ endowed with the weak* topology. We generalize this as follows.

Main Result

Let E be a locally convex space. Denote by E' the topological dual space of E and set $E'_{w^*} := \big(E', \sigma(E', E)\big)$, where $\sigma(E', E)$ is the weak* topology on E'.

Let X be a Tychonoff space. A map $T : X \to E'$ is called *weak* continuous* if T is continuous as a map from T to E'_{w^*}. Any map T from X into the algebraic dual E^* of E defines an *associative map* $T_E : X \times E \to \mathbb{F}$ by

$$T_E(x, z) := \langle T(x), z\rangle \quad \text{where } \ x \in X \ \text{ and } \ z \in E.$$

We need the following lemma.

Lemma 6: *A map $T : X \to E'$ is weak* continuous if, and only if, the associative map T_E is linear by the second argument and separately continuous.*

Main Result

Lemma 6 shows that the following notions are well-defined.

Definition 7: Let X be a Tychonoff space, and let E be a locally convex space. We shall say that a map $T : X \to E'$ is *almost k-compact* (resp., *almost k-sequential*) if it is weak* continuous and there are a neighborhood U of zero in E and a compact subset (resp., a null sequence) K of $C_k(X)$ such that the family

$$\{T_E(x,a) : \; a \in U\} \subseteq C_k(X)$$

is contained in the absolutely convex closed hull $\overline{\mathrm{acx}}(K)$ of K.

We add to the definition "*almost*" because the absolutely convex closed hull $\overline{\mathrm{acx}}(K)$ of K need not be compact. Locally convex spaces in which the absolutely convex closed hull of each compact subset is compact are said to have the *convex compactness property* (ccp).

Main Result

Now we are ready to formulate the announced Banach space characterization of (sequentially) Ascoli spaces.

Theorem 8: *For a Tychonoff space X, the following assertions are equivalent:*

(i) *X is an Ascoli (resp., sequentially Ascoli) space;*

(ii) *for each cardinal Γ, every k-continuous and almost k-compact (resp., almost k-sequential) map $T : X \to \ell_\infty(\Gamma)$ is continuous;*

(iii) *for each Banach space E, every k-continuous and almost k-compact (resp., almost k-sequential) map $T : X \to E'_\beta$ is continuous.*

Numerous characterizations of (sequentially) Ascoli spaces and $k_{\mathbb{R}}$-spaces will be published (I hope) soon.

An Open Problem

I want to finish my talk with the following open problem. Let us recall

Theorem 5: *A Tychonoff space X is sequentially Ascoli if and only if every k-continuous bounded map $T : X \to c_0$ is continuous whenever it is continuous as a map from X into $(c_0)_w$.*

Problem 1: Is it true that a Tychonoff space X is (sequentially) Ascoli if and only if for each Banach space E, every k-continuous bounded map $T : X \to E$ is continuous whenever it is continuous as a map from X into E_w.

Problem 2: Is it true that a Tychonoff space X is (sequentially) Ascoli if and only if for each compact space K, every k-continuous bounded map $T : X \to C(K)$ is continuous whenever it is continuous as a map from X into $C(K)_w$.

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Thank you!