Dynamics of operators on the space of Radon measures

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If $\mathcal{X}$ is a Banach space, the set of all bounded linear operators from $\mathcal{X}$ into $\mathcal{X}$ is denoted by $B(\mathcal{X})$. Also, we denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Definition

Let $\mathcal{X}$ be a Banach space. A sequence $(T_n)_{n\in\mathbb{N}_0}$ of operators in $B(\mathcal{X})$ is called *topologically transitive* if for each non-empty open subsets U, V of $\mathcal{X}$, $T_n(U) \cap V \neq \varnothing$ for some $n \in \mathbb{N}$.
A single operator T in $B(\mathcal{X})$ is called topologically transitive if the sequence $(T^n)_{n\in\mathbb{N}_0}$ is topologically transitive.Similarly, we say that T is *topologically semi-transitive* on X if for each pair of open non-empty subsets O_1 and O_2 of X there exists some $n \in \mathbb{N}$ and some $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda T^n(O_1) \cap O_2 \neq \varnothing$. We say that T is *topologically Cesáro hyper-transitive* on X if for each pair of open non-empty subsets O_1 and O_2 of X there exists a strictly increasing sequence of natural numbers $\{n_k\}_k$ such that $n_k^{-1} T^{n_k}(O_1) \cap O_2 \neq \varnothing$ for all k.

Definition

[**tsi**] Let X be a topological space. Let $\alpha : X \longrightarrow X$ be invertible, and α, α^{-1} be Borel measurable. We say that α is *aperiodic* if for each compact subset K of X, there exists a constant $N > 0$ such that for each $n \geq N$, we have $K \cap \alpha^n(K) = \varnothing$, where α^n means the n-fold combination of α.

We let Ω be a locally compact non-compact Hausdorff space and α be an aperiodic homeomorphism of Ω. As usual, $C_0(\Omega)$ denotes the space of all continuous functions on Ω vanishing at infinity, equipped with the supremum norm. Moreover, we let w be a positive continuous bounded function on Ω such that also w^{-1} is also bounded and we put then $T_{\alpha,w}$ to be the weighted composition operator on $C_0(\Omega)$ with respect to α and w, that is $T_{\alpha,w}(f) = w \cdot (f \circ \alpha)$ for all $f \in C_0(\Omega)$. Easily, one can see that by the above assumptions $T_{\alpha,w}$ is well-defined and $\|T_{\alpha,w}\| \leq \|w\|_{\mathrm{sup}}$. Since $\frac{1}{w}$ is also bounded, then $T_{\alpha,w}$ is invertible and we have

$$T_{\alpha,w}^{-1} f = \frac{f \circ \alpha^{-1}}{w \circ \alpha^{-1}}, \quad (f \in C_0(\Omega)).$$

Simply we denote $S_{\alpha,w} := T_{\alpha,w}^{-1}$.

Remark

If w and $\frac{1}{w}$ are weights, the inverse of a weighted composition operator $T_{\alpha,w}$ is also a weighted composition operator. In fact, $S_{\alpha,w} = T_{\alpha^{-1},\frac{1}{w\circ\alpha^{-1}}}$. Moreover, if T_{α_1,w_1} and T_{α_2,w_2} are two weighted composition operators, then

$$T_{\alpha_2,w_2} \circ T_{\alpha_1,w_1} = T_{\alpha_1\circ\alpha_2,w_2(w_1\circ\alpha_2)},$$

so the composition of two weighted composition operators is again a weighted composition operator. By some calculation one can see that for each $n \in \mathbb{N}$ and $f \in C_0(\Omega)$,

$$T^n_{\alpha,w} f = \left(\prod_{j=0}^{n-1} (w \circ \alpha^j)\right) \cdot (f \circ \alpha^n) \ (1)$$

and

$$S^n_{\alpha,w} f = \left(\prod_{j=1}^{n} (w \circ \alpha^{-j})\right)^{-1} \cdot (f \circ \alpha^{-n}). \ (2)$$

Lemma

The following are equivalent.

(i) $T_{\alpha,w}$ is topologically transitive on $C_0(\Omega)$.

(ii) For every compact subset K of Ω there exists a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that

$$\lim_{k\to\infty}(\sup_{t\in K}|\prod_{j=0}^{n_k-1}(w\circ\alpha^{j-n_k})(t)|) = \lim_{k\to\infty}(\sup_{t\in K}|\prod_{j=0}^{n_k-1}(w\circ\alpha^{j})^{-1}(t)|) = 0.$$

The adjoint $T^*_{\alpha,w}$ is a bounded operator on $M(\Omega)$ where $M(\Omega)$ stands for the Banach space of all regular Borel measures on Ω equipped with the total variation norm. It is straightforward to check that

$$T^*_{\alpha,w}(\mu)(E) = \int_E w \circ \alpha^{-1} \, d\mu \ \circ \alpha^{-1}$$

for every $\mu \in M(\Omega)$, and every measurable subset E od Ω. By (1) and (2) it follows then that for every $n \in \mathbb{N}$, $\mu \in M(\Omega)$ and a Borel measurable subset $E \subseteq \Omega$ we have that

$$T^{*n}_{\alpha,w}(\mu)(E) = \int_E \prod_{j=0}^{n-1} w \circ \alpha^{j-n} \, d\mu \ \circ \alpha^{-n}$$

and

$$S^{*n}_{\alpha,w}(\mu)(E) = \int_E \prod_{j=1}^{n} (w \circ \alpha^{n-j})^{-1} \, d\mu \ \circ \alpha^{n}.$$

Proposition

The following statements are equivalent.
i) $T^*_{\alpha,w}$ is topologically transitive on $M(\Omega)$.
ii) For every compact subset K of Ω and any two measures μ, v in $M(\Omega)$ with $|v|(K^c) = |\mu|(K^c) = 0$ there exist a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ and sequences $\{A_k\}$, $\{B_k\}$ of Borel subsets of K such that $\alpha^{n_k}(K) \cap K = \varnothing$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} |\mu|(A_k) = \lim_{k\to\infty} |v|(B_k) = 0,$$

$$\lim_{k\to\infty} \sup_{t\in K\cap A_k^c} \Big(\prod_{j=0}^{n_k-1} (w\circ\alpha^j)(t)\Big) = \lim_{k\to\infty} \sup_{t\in K\cap B_k^c} \Big(\prod_{j=1}^{n_k} (w\circ\alpha^{-j})^{-1}(t)\Big) = 0.$$

Corollary

We have that ii) $\Rightarrow$ i)
i) $T^*_{\alpha,w}$ is topologically transitive on $M(\Omega)$.
ii) For every compact subset K of Ω there exists a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that

$$\lim_{k\to\infty} \sup_{t\in K}\left(\prod_{j=0}^{n_k-1}(w\circ\alpha^j)(t)\right) = \lim_{k\to\infty} \sup_{t\in K}\left(\prod_{j=1}^{n_k}(w\circ\alpha^{-j})^{-1}(t)\right) = 0.$$

Open problem

Does there exist an example where the equivalent conditions of the part ii) in the previous proposition are satisfied, whereas the sufficient conditions of the part ii) in this corollary are not satisfied ?

Example

Let $\Omega = \mathbb{R}, \ \alpha(t) = t + 1$ for all $t \in \mathbb{R}$ and

$$w(t) = \begin{cases} 2 \ \text{ for } t \leq -1, \\ \dfrac{1}{2} \ \text{ for } t \geq 1, \\ \text{linear on the segment } [-1, 1]. \end{cases}$$

In this case, the sufficient conditions of the preceding corollaries are satisfied.

In general, if $M, \epsilon > 0$ such that $1 + \epsilon < M$, $1 - \epsilon > \dfrac{1}{M}$, and $K_1, K_2 > 0$, then if $w \in C_b(\mathbb{R})$ satisfies that $M \geq |w(t)| \geq 1 + \epsilon$ for all $t \leq -K_1$ and $\dfrac{1}{M} \leq |w(t)| \leq 1 - \epsilon$ for all $t \geq K_2$, then the sufficient conditions of the preceding corollary are satisfied.

Proposition

The following statements are equivalent.

i) $T^*_{\alpha,w}$ is topologically semi-transitive on $M(\Omega)$.

ii) For every compact subset K of Ω and any two measures μ, v in $M(\Omega)$ with $|\mu|(K^c) = |v|(K^c) = 0$ there exist a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ and sequences $\{A_k\}_k, \{B_k\}_k$ of Borel subsets of K such that $\alpha^{n_k}(K) \cap K = \varnothing$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} |\mu|(A_k) = \lim_{k\to\infty} |v|(B_k) = 0,$$

$$\lim_{k\to\infty} [\left(\sup_{t\in A_k^c\cap K} \prod_{j=0}^{n_k-1} \left(w\circ\alpha^j\right)(t)\right) \cdot \left(\sup_{t\in B_k^c\cap K} \prod_{j=1}^{n_k} \left(w\circ\alpha^{-j}\right)^{-1}(t)\right)] = 0.$$

If α is not aperiodic, then we only have $ii) \Rightarrow i)$.

Proposition

The following statements are equivalent.

i) $T^*_{\alpha,w}$ is topologically Cesáro hyper-transitive on $M(\Omega)$.

ii) For every compact subset K of Ω and any two measures μ, v in $M(\Omega)$ with $|\mu|(K^c) = |v|(K^c) = 0$ there exist a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ and sequences $\{A_k\}_k, \{B_k\}_k$ of Borel subsets of K such that $\alpha^{n_k}(K) \cap K = \varnothing$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} |\mu|(A_k) = \lim_{k\to\infty} |v|(B_k) = 0,$$

$$\lim_{k\to\infty} \left(\sup_{t \in A_k^c \cap K} n_k^{-1} \prod_{j=0}^{n_k-1} \left(w \circ \alpha^j\right)(t) \right) =$$

$$= \lim_{k\to\infty} \left(\sup_{t \in B_k^c \cap K} n_k \prod_{j=1}^{n_k} \left(w \circ \alpha^{-j}\right)^{-1}(t) \right) = 0.$$

If α is not aperiodic, then we only have $ii) \Rightarrow i)$.

Example

Let $\Omega = \mathbb{R}$, $\alpha : \mathbb{R} \to \mathbb{R}$ be given by $\alpha(t) = t + 1$ for all $t \in \mathbb{R}$ and w be a continuous bounded positive function on $\mathbb{R}$. If there exist some $M, \delta, K_1, K_2 > 0$ such that $1 < M - \delta \leq w(t) \leq M$ for all $t \leq -K_1$ and $w(t) = 1$ for all $t \geq K_2$, then $T^*_{\alpha,w}$ is topologically Cesáro hyper-transitive, but it is not topologically transitive on $M(\Omega)$. On the other hand, if $\alpha(t) = t - 1$ for all $t \in \mathbb{R}$, $\frac{1}{M} \leq w(t) \leq \frac{1}{M-\delta}$ for all $t \leq -K_1$ and $1 = w(t)$ for all $t \geq K_2$, then $T^*_{\alpha,w}$ is topologically semi-transitive, but it is neither topologically Cesáro hyper-transitive nor topologically transitive on $M(\Omega)$.

For each $n \in \mathbb{N}$, we set now $C^{*(n)}_{\alpha,W} = \frac{1}{2}(T^{*n}_{\alpha,W} + S^{*n}_{\alpha,W})$.

Proposition

We have that $(ii) \Rightarrow (i)$:

(i) The sequence $(C^{*^{(n)}}_{\alpha,w})$ is topologically transitive on $M(\Omega)$.

(ii) For every compact subset K od Ω and any two measures μ, v in $M(\Omega)$ with $|\mu|(K^c) = |v|(K^c) = 0$ there exist a strictly increasing sequence $\{n_k\} \subseteq \mathbb{N}$ and sequences $\{A_k\}_k, \{F_k\}_k, \{D_k\}_k$ of Borel subsets of K such that

$$\lim_{k\to\infty} |\mu|(A_k) = \lim_{n\to\infty} |v|(A_k) = 0,$$

$$\lim_{k\to\infty} \sup_{t\in K\cap A_k^c} \left(\prod_{j=0}^{n_k-1}(w\circ\alpha^j)(t)\right) = \lim_{k\to\infty} \sup_{t\in K\cap A_k^c} \left(\prod_{j=0}^{n_k-1}(w\circ\alpha^{-j})^{-1}(t)\right) = 0,$$

$$\lim_{k\to\infty} \sup_{t\in F_k} \left(\prod_{j=0}^{2n_k-1}(w\circ\alpha^j)(t)\right) = \lim_{k\to\infty} \sup_{t\in D_k} \left(\prod_{j=1}^{2n_k}(w\circ\alpha^{-j})^{-1}(t)\right) = 0,$$

where $F_k \cap D_k = \varnothing$ and $A_k^c \cap K = F_k \cup D_k$ for all k.

Corollary

We have that $(ii) \Rightarrow (i)$:

(i) *The sequence* $(C^{*^{(n)}}_{\alpha,w})$ *is topologically transitive on* $M(\Omega)$.

(ii) *For every compact subset* K *od* Ω *we have that*

$$\lim_{n\to\infty} \sup_{t\in K} \left(\prod_{j=0}^{n-1}(w\circ\alpha^{j})(t)\right) = \lim_{n\to\infty} \sup_{t\in K} \left(\prod_{j=0}^{n-1}(w\circ\alpha^{-j})^{-1}(t)\right) = 0,$$

$$\lim_{n\to\infty} \sup_{t\in K} \left(\prod_{j=0}^{2n-1}(w\circ\alpha^{j})(t)\right) = \lim_{n\to\infty} \sup_{t\in K} \left(\prod_{j=1}^{2n}(w\circ\alpha^{-j})^{-1}(t)\right) = 0.$$

Let Ω be a compact Hausdorff space and $M_r(\Omega)$ denote the space of all signed Radon measures on Ω with the norm $\| v \| = |v|(\Omega)$. For $v \in M_r(\Omega)$, let $\phi_v : C_{\mathbb{R}}(\Omega) \to \mathbb{R}$ be defined by

$$\phi_v(f) = \int_\Omega f dv.$$

Then $v \to \phi_v$ is an isometric isomorphism of $(M_r, (\Omega), \| \cdot \|)$ onto $((C_{\mathbb{R}}(\Omega))^*, \| \cdot \|_\infty)$.

Let $k : \Omega \times \Omega \to \mathbb{R}$ be a continuous non - negative function and let μ be a positive Radon measure on Ω such that

$$\int_{\Omega} k(x,y)d\mu(y) > 0 \text{ for all } x \in \Omega.$$

Put then $\tilde{k} : \Omega * \Omega \to \mathbb{R}$ to be defined as

$$\tilde{k}(x,y) = \frac{k(x,y)}{\int_{\Omega} k(x,y)d\mu(y)}.$$

Then $\tilde{k}$ is continuous, nonnegative and $\int_{\Omega} \tilde{k}(x,y)d\mu(y) = 1$.

We consider now the integral operator T_k on $C_{\mathbb{R}}(\Omega)$ given by

$$T_k(f)(x) = \int_{\Omega} \tilde{k}(x,y)f(y)d\mu(y) \text{ for all } x \in \Omega.$$

The adjoint of T_k, will be denoted by T_k^*.

Proposition

Under the above assumptions, if $||\tilde{k}||_\infty < \frac{2}{\mu(\Omega)}$, then there exists a unique invariant probability Radon measure $\tilde{v}$ on Ω such that

$$|(T_k^*)^n(v) - \tilde{v}|\ (\Omega) \to 0 \text{ as } n \to \infty$$

for all probability Radon measures v on Ω.

As a concrete example, let $\Omega = [0, 2\pi],\ \mu$ be the Lebesgue measure on $[0, 2\pi]$ and $k : \Omega\ \times\ \Omega \to \mathbb{R}$ be given as $k(x, y) = \frac{1}{4} sin(\frac{1}{4}(x + y))$.

We recall that a subset S of a Banach space Y is called *spaceable* in Y if $S \cup \{0\}$ contains a closed infinite-dimensional subspace of Y.
In this presentation, a subset B of a vector space Y is called a cone if for each scalar c, $cB \subseteq B$.
For a Borel measurable subset E and some $\mu \in M(\Omega)$, we let μ_E be the measure given by $\mu_E(B) := \mu(B \cap E)$ for every Borel subset B of Ω. If K is a cone in $M(\Omega)$, we denote

$$\widetilde{K} := \{\mu_E : \ \mu \in K, \ E \text{ is Borel}\}.$$

Since for every scalar $\lambda \in \mathbb{C}$ we have $(\lambda\mu)_E = \lambda\, \mu_E$, it follows that $\widetilde{K}$ is a cone. Moreover, $K \subseteq \widetilde{K}$.

Proposition

Let K be a cone in $M(\Omega)$. If there exists a sequence of mutually disjoint Borel subsets $\{E_n\}_{n\in\mathbb{N}}$ of Ω such that for all n

$$\{\mu_{E_n} : \mu \in \widetilde{K}\} \neq \{\mu_{E_n} : \mu \in M(\Omega)\},$$

then $M(\Omega) \setminus \tilde{K}$ (and consequently $M(\Omega) \setminus K$) is spaceable in $M(\Omega)$.

Corollary

Let K be the cone of all scalar multiples of positive Radon measures on a non-compact, locally compact Hausdorff space Ω. Then $M(\Omega) \setminus K$ is spaceable in $M(\Omega)$.

Thank you for attention !

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