Numerical moduli in the geometry of [special] multi-flags

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Singularity Classes of Special 2-Flags*

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Abstract. In the paper we discuss certain classes of vector distributions in the tangent bundles to manifolds, obtained by series of applications of the so-called generalized Cartan prolongations (gCp). The classical Cartan prolongations deal with rank-2 distributions and are responsible for the appearance of the Goursat distributions. Similarly, the so-called special multi-flags are generated in the result of successive applications of gCp's. Singularities of such distributions turn out to be very rich, although without functional moduli of the local classification. The paper focuses on special 2-flags, obtained by sequences of gCp's applied to rank-3 distributions. A stratification of germs of special 2-flags of all lengths into *singularity classes* is constructed. This stratification provides invariant geometric significance to the vast family of local polynomial pseudo-normal forms for special 2-flags introduced earlier in [Mormul P., *Banach Center Publ.*, Vol. 65, Polish Acad. Sci., Warsaw, 2004, 157–178]. This is the main contribution of the present paper. The singularity classes endow those multi-parameter normal forms, which were obtained just as a by-product of sequences of gCp's, with a geometrical meaning.

Key words: generalized Cartan prolongation; special multi-flag; special 2-flag; singularity class

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1 Introduction and main theorem

The aim of the current paper is to present a new and rather rich stratification of singularities of (special) 2-flags which naturally generalize 1-flags. Before doing that, it will be useful to briefly recall 1-flags and their singularities. These are, in the contemporary terminology, rank-2 and corank ≥ 2 subbundles $D \subset TM$ in the tangent bundle to a smooth manifold M, together with the tower of consecutive *Lie squares* $D \subset [D, D] \subset [[D, D], [D, D]] \subset \cdots$ satisfying the property that the linear dimensions of tower's members are $2, 3, 4, \ldots$ at *every* point in M. (In $(\dim M - 2)$ steps the tower reaches the full tangent bundle TM.) These objects had emerged in the papers [7, 21, 6] and were later popularized in a book by Goursat in the 1920s. In the result, such distributions D are now called the *Goursat distributions*, or sometimes the *Cartan–Goursat distributions.* The respective flags are called the *Goursat flags.* Although this definition is quite restrictive, still such flags exist in all lengths. Indeed, for every $s \geq 2$, the canonical contact system $\mathcal{C}^s$ (the jet bundle or the Cartan distribution in the terminology of [9]) on the jet space $J^s(1, 1)$ is a Goursat distribution of corank s; its flag has length s. However, each distribution $\mathcal{C}^s$ is homogeneous because its germs at every two points are equivalent by a local diffeomorphism of $J^s(1, 1)$. Therefore, these contact systems have no singularities. It should be noted that nowadays the contact systems on $J^s(1, 1)$ are *also* known under the name 'Goursat normal forms' and are characterized as such in [3] (Theorem 5.3 in Chapter II).

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Special 2-flags in lengths not exceeding four: a study in strong nilpotency of distributions

Piotr Mormul* and Fernand Pelletier

Abstract

In the recent years, a number of issues concerning distributions generating 1-flags (called also Goursat flags) has been analyzed. Presently similar questions are discussed as regards distributions generating *multi*-flags. (In fact, only so-called *special* multi-flags, to avoid functional moduli.) In particular and foremost, special 2-flags of small lengths are a natural ground for the search of generalizations of theorems established earlier for Goursat objects. In the present paper we locally classify, in both C^ω and C^∞ categories, special 2-flags of lengths not exceeding four. We use for that the known facts about special multi-flags along with fairly recent notions like *strong nilpotency* of distributions. In length four there are already 34 orbits, the number to be confronted with only 14 singularity classes – basic invariant sets discovered in 2003.

As a common denominator for different parts of the paper, there could serve the fact that only rarely multi-flags' germs are strongly nilpotent, whereas all of them are weakly nilpotent, or nilpotentizable (possessing a local nilpotent basis of sections).

1 Definition of special k-flags and their singularities

Special k-flags (the natural parameter $k \geq 2$ is sometimes called 'width') of lengths $r \geq 1$ can be defined in several equivalent ways, like in [KRub], [PaR], [M2]. All these approaches can be reduced to one transparent definition. (The reduction is via two early Bryant's results from [B], one lemma from [PaR], and the answer to a recent question of Zhitomirskii, cf. p. 165 in [M2].)

Namely, for a distribution D on a manifold M, the tower of consecutive Lie squares of D

$$D = D^r \subset D^{r-1} \subset D^{r-2} \subset \cdots \subset D^1 \subset D^0 = TM$$

(that is, $[D^j, D^j] = D^{j-1}$ for $j = r, r-1, \ldots, 2, 1$) should consist of distributions of ranks, starting from the smallest object D^r: $k+1,\ 2k+1,\ \ldots,\ rk+1,\ (r+1)k+1 = \dim M$ such that

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To our knowledge the general theory behind Goursat multi-flags made their first appearance in the works of A. Kumpera and J.L. Rubin [11]. P. Mormul has also been very active in breaking new ground [15], and developed new combinatorial tools to investigate the normal forms of these distributions. Our work is founded on a recent article [22] by Shibuya and Yamaguchi that demonstrates a universality result which essentially states that any Goursat multi-flag arises as a type of lifting of the tangent bundle of $\mathbb{R}^n$.

In this paper we concentrate on the problem of classifying Goursat multi-flags of small length. Specifically, we will consider Goursat 2-flags of length up to 4. Goursat 2-flags exhibit many new geometric features our old Goursat 1-flags did not possess [19].

Our main result states that there are 34 inequivalent Goursat 2-flags of length 4 and we provide the exact number of Goursat 2-flags for each length $k \leqslant 3$ as well.

Our approach is constructive. Due to space limitations we will write down only a few instructive examples.

In [22] Shibuya and Yamaguchi establish that every Goursat 2-flag germ appears somewhere within the following tower of fiber bundles:

$$\cdots \to \mathcal{P}^4(2) \to \mathcal{P}^3(2) \to \mathcal{P}^2(2) \to \mathcal{P}^1(2) \to \mathcal{P}^0(2) = \mathbb{R}^3, \tag{1}$$

and the fiber of the projection map from $\mathcal{P}^k(2)$ to $\mathcal{P}^{k-1}(2)$ is a real projective plane, and adding the dimensions one obtains the dimension formula $\dim(\mathcal{P}^k(2)) = 3 + 2k$.

Each manifold $\mathcal{P}^k(2)$ is equipped with a rank 3 nonholonomic distribution Δ_k, and there is a simple geometric relation between the distributions pertaining to neighboring levels. The construction of Δ_k is recursive, and depends upon the geometric data at the base level $\mathcal{P}^0$.

The distributions Δ_k in $\mathcal{P}^k(2)$ are themselves Goursat 2-flags of length k. Moreover, two Goursat 2-flags are equivalent if and only if the corresponding points of the Monster Tower are mapped one to the other by a *symmetry* of the tower at level k. The paper [22] also establishes that all such symmetries are prolongations of diffeomorphisms of $\mathbb{R}^3$. The above observations tell us that

the classification problem for Goursat 2-flags is equivalent to the classification of points within the Monster Tower up to symmetry.

In order to solve this latter problem we use a combination of two methods, namely the singular curve method as in [18] and a new method that we call the *isotropy method*. A variant of the isotropy method was already used in [18], and it is somewhat inspired by É. Cartan's *moving frame* method [8].

We would like to mention that P. Mormul and Pelletier [17] have proposed an alternative solution to the classification problem. In their classification work, they employed Mormul's results and tools that came from his recent work with Goursat n-flags. In [16], Mormul discusses two coding systems for special 2-flags and showed that the two coding systems are the same. One system is the *extended Kumpera–Ruiz system*, which is a coding system used to describe 2-flags. The other is called *singularity class coding*, which is an intrinsic coding system that describes the sandwich diagram [18] associated to 2-flags. A brief outline on how these coding systems relate to the RVT coding is discussed in [6]. Then, building upon Mormul's work in [14], Mormul and Pelletier used the idea of strong nilpotency of special multi-flags, along with the properties of his two coding systems, to classify these distributions up to length 4. Our 34 orbits agree with theirs.

In Section 2 we acquaint ourselves with the main definitions necessary for the statements of our main results, and a few explanatory remarks to help the reader progress through the theory with us. Section 3 consists of the statements of our main results. In Section 4 we discuss the basic tools and ideas that will be needed to prove our various results. Section 5 is devoted to technicalities and the actual proofs. Finally, in Section 6, we provide a quick summary of our findings and other questions to pursue concerning the Monster Tower.

For the record, we have also included Appendix A where our lengthy computations are contained.

2. Preliminaries and main definitions

A *geometric distribution* hereafter denotes a linear subbundle of the tangent bundle with fibers of constant dimension.

2.1. Prolongation

Let the pair (Z, Δ) denote a manifold Z of dimension d equipped with a distribution Δ of rank r. We denote by $\mathbb{P}(\Delta)$ the *projectivization* of Δ. As a manifold,

$$\mathbb{P}(\Delta) \equiv Z^1$$

has dimension $d + (r - 1)$.

Example 2.1. Take $Z = \mathbb{R}^3$, $\Delta = TR^3$ viewed as a rank 3 distribution. Then Z^1 is simply the trivial bundle $\mathbb{R}^3 \times \mathbb{P}^2$, where the factor on the right denotes the projetive plane.

The number of different singularity classes of special 2-flags of length $r \geq 3$ is

$$2 + 3 + 3^2 + \cdots + 3^{r-2} \;=\; \frac{1 + 3^{r-1}}{2}\,. \tag{1}$$

(One focuses attention on the position of the first letter 2 in the class' code, remembering that the codes satisfy the least upward jumps rule: no letter 2 or else that letter at the very end – account for the summand 2, that letter at the one before last position accounts for the summand 3, and so on. Then that letter at the second position accounts for the biggest summand 3^{r-2}.)

1.5 Moduli among parameters in pseudo-normal forms.

Once the singularity classes (in the present paper – only for 2-flags) and faithful to them pseudo-normal forms EKR have been recalled, one of the first imposing questions is that about the *status* of real parameters entering the EKR forms. The same question concerning parameters in normal forms for germs of 1-flags, sparked by the benchmark work [KRui], had remained without answer over a considerable period 1982–97.

With examples of moduli of 1-flags at hand, it is not long to produce an example of an EKR parameter that is a true modulus. To this end, choose the following family of EKR's **1.2.1.2.1.2.1** sitting (see Theorem 2) in the singularity class 1.2.1.2.1.2.1:

$$\begin{aligned}
dx_1 - x_2 dt &= 0 & \qquad dy_1 - y_2 dt &= 0\\
dt - x_3 dx_2 &= 0 & dy_2 - y_3 dx_2 &= 0\\
dx_3 - (1 + x_4) dx_2 &= 0 & dy_3 - y_4 dx_2 &= 0\\
dx_2 - x_5 dx_4 &= 0 & dy_4 - y_5 dx_4 &= 0\\
dx_5 - (1 + x_6) dx_4 &= 0 & dy_5 - y_6 dx_4 &= 0\\
dx_4 - x_7 dx_6 &= 0 & dy_6 - y_7 dx_6 &= 0\\
dx_7 - (c + x_8) dx_6 &= 0 & dy_7 - y_8 dx_6 &= 0\,,
\end{aligned} \tag{2}$$

where $c \in \mathbb{R}$ is an arbitrary real parameter and these objects are considered as germs at $0 \in \mathbb{R}^{17}(t,\, x_1,\, y_1, \ldots,\, x_8,\, y_8)$. (Due to the Pfaffian equations' description, it is not instantly visible that the objects sit in an EKR. Yet, by the time we prove the statement in Appendix (Section 8), it will be clear that the proposed objects belong to a concrete EKR class of normal forms). The proof is being postponed to keep the exposition balanced.

Remark 2. (a) The 1-parameter family in (2) is, as it stands, written for the width $k = 2$ (there are only two columns of Pfaffian equations). However, a similar family could be proposed for any bigger width. The reader can easily figure out the potential 3rd, ..., kth columns, all constructed on the pattern of the second column, with no additional constants (the non-zero constants, decisive for the example, always in the first column only). The proof for the analogous objects inside the EKR class **1.2.1.2.1.2.1** in the space of special k-flags, $k > 2$, would be essentially the same, only the basic vector equation would be longer and so would be equations on the levels X_5 and X_3.

length	# sing classes	# RV classes	# orbits
2	2	2	2
3	5	6	7
4	14	23	34
5	41	98	?
6	122	433	??
7	365	1935	∞

Question. How to partition a given singularity class of special 2-flags into (much finer!) RV classes of [4] ? And, all the more so, for special m-flags, $m > 2$?!

5.1. **Singularity classes of special 2-flags refining the sandwich classes.** We first divide all existing germs of special 2-flags of length r into 2^{r-1} pairwise disjoint *sandwich classes* in function of the geometry of the distinguished spaces in the sandwiches (at the reference point for a germ) in Sandwich Diagram on p. 3, and label those aggregates of germs by words of length r over the alphabet $\{1, \underline{2}\}$ starting (on the left) with 1, having the second cipher $\underline{2}$ iff $D^2(p) \subset F(p)$, and for $3 \le j \le r$ having the j-th cipher $\underline{2}$ iff $D^j(p) \subset L(D^{j-2})(p)$. More details about the sandwich classes are given in section 1.2 in [18].

This construction puts in relief possible non-transverse situations in the sandwiches. For instance, the second cipher is $\underline{2}$ iff the line $D^2(p)/L(D^1)(p)$ is not transverse, in the space $D^1(p)/L(D^1)$, to the codimension one subspace $F(p)/L(D^1)(p)$, and similarly in further sandwiches. This resembles very much the KR-classes of Goursat germs constructed in [11]. In length r the number of sandwiches has then been $r-2$ (and so the # of KR classes 2^{r-2}). For 2-flags the number of sandwiches is $r-1$ because the covariant distribution of D^1 comes into play and gives rise to one additional sandwich.

Passing to the main construction underlying our present contribution, we refine further the singularities of special 2-flags and recall from [15] how one passes from the sandwich classes to *singularity classes.* In fact, to any germ $\mathcal{F}$ of a special 2-flag associated is a word $\mathcal{W}(\mathcal{F})$ over the alphabet $\{1, 2, 3\}$, called the 'singularity class' of $\mathcal{F}$. It is a specification of the word 'sandwich class' for $\mathcal{F}$ (this last being over, reiterating, the alphabet $\{1, \underline{2}\}$) with the letters $\underline{2}$ replaced either by 2 or 3, in function of the geometry of $\mathcal{F}$.

In the definition that follows we keep fixed the germ of a rank-3 distribution D at $p \in M$, generating on M a special 2-flag $\mathcal{F}$ of length r.

Suppose that in the sandwich class $\mathcal{C}$ of D at p there appears somewhere, for the first time when reading from the left to right, the letter $\underline{2} = j_m$ (j_m is, as we know, not the first letter in $\mathcal{C}$) **and** that there are in $\mathcal{C}$ other letters $\underline{2} = j_s$, $m < s$, as well. We will specify each such j_s to one of the two: 2 or 3. (The specification of that first $j_m = \underline{2}$ will be made later and will be trivial.) Let the nearest $\underline{2}$ standing to the left to j_s be $\underline{2} = j_t$, $m \le t < s$. These two 'neighbouring' letters $\underline{2}$ are separated in $\mathcal{C}$ by $l = s - t - 1 \ge 0$ letters 1.

The gist of the construction consists in taking the *small flag* of precisely original flag's member D^s,

$$D^s = V_1 \subset V_2 \subset V_3 \subset V_4 \subset V_5 \subset \cdots,$$

$V_{i+1} = V_i + [D^s, V_i]$, then focusing precisely on this new flag's member V_{2l+3}. Reiterating, in the t-th sandwich, there holds the inclusion: $F(p) \supset D^2(p)$ when $t = 2$, or else $L(D^{t-2})(p) \supset D^t(p)$ when $t > 2$. This serves as a preparation to our punch line (cf. [15, 17]).

Surprisingly perhaps, specifying j_s to 3 goes via replacing D^t by V_{2l+3} in the relevant sandwich inclusion at the reference point. That is to say, $j_s = \underline{2}$ is being specified to 3 if and only if $F(p) \supset V_{2l+3}(p)$ (when $t = 2$) or else $L(D^{t-2})(p) \supset V_{2l+3}(p)$ (when $t > 2$) holds.

The one before last slide is from [JoS 21 (2020)] – the infinitesimal symmetries offered a chance to advance the local classification problem, at least for special 2-flags.

Yet they were given recursively, step after step (or: level after level). Recently **Andrzej Weber** implemented those recurrencies in Wolfram Mathematica and that lit the orange light for the solution of the problem.

The last slide words the result concerning the singularity classes of special 2-flags at level 5 (or: in length 5).

↖ encoded by the words over $\{1,2,3\}$ starting with 1 and such that the first 2 (if any) goes before the first 3 (if any), of length 5

1.1.1.1.1 ← generic open dense stratum, homogeneous, the local geometry of $C \subset TJ^5(1,2)$.

1.1.1.1.2

~~1.1.1.1.3~~

. . . . . .

Will illustrate the technique of proof

! { 1.2.1.2.1 ← on this class, the grandfather of the pioneering class 1.2.1.2.1.2.1

1.2.2.1.2

1.2.3.1.2 }

The singularity 1.2.1.2.1 is properly viewed (only) in the EKR **1.2.1.2.1** ← these glasses are in use till the end

The points $p_c = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\1\\b=0\\c \end{bmatrix}$ sitting in 1.2.1.2.1 will turn out to be pairwise inequivalent.

The mentioned orange light first: An arbitrary i.s. has at p_c the components:

$$\begin{bmatrix} A\\B\\C \end{bmatrix} = \begin{bmatrix} 0\\0\\0 \end{bmatrix}$$

$$\begin{bmatrix} F^1\\G^1 \end{bmatrix} = \begin{bmatrix} B^{100}\\C^{100} \end{bmatrix} = \begin{bmatrix} 0\\0 \end{bmatrix}$$

$$\begin{bmatrix} F^2\\G^2 \end{bmatrix} = \begin{bmatrix} 0\\C^{010} \end{bmatrix} = \begin{bmatrix} 0\\0 \end{bmatrix}$$

$$\begin{bmatrix} F^3\\G^3 \end{bmatrix} = \begin{bmatrix} 3A^{100}-2B^{010}\\C^{200} \end{bmatrix} = \begin{bmatrix} 0\\0 \end{bmatrix}$$

$$\begin{bmatrix} F^4\\G^4 \end{bmatrix} = \begin{bmatrix} 0\\-2A^{100}+C^{001}+C^{200} \end{bmatrix} = \begin{bmatrix} 0\\0 \end{bmatrix}$$

$$\begin{bmatrix} F^5\\G^5 \end{bmatrix} = \begin{bmatrix} 0\\c(-5A^{100}+2B^{010}+C^{001}) \end{bmatrix} = \begin{bmatrix} 0\\c(G^4-G^3-F^3) \end{bmatrix}$$

A, B, C are (free) base components: $A\partial_t + B\partial_{x0} + C\partial_{y0}$, superscripts are partial derivatives at $0 \in \mathbb{R}^3(t, x0, y0)$,

for inst. $C^{200} = \frac{\partial^2 C}{\partial t^2}\Big|_{(0,0,0)}$, $B^{010} = \frac{\partial B}{\partial x0}\Big|_{(0,0,0)}$, $C^{001} = \frac{\partial C}{\partial y0}\Big|_{(0,0,0)}$.

We stipulate the vanishing of all components got at previous steps Nº 0, 1, 2, 3, 4. And focus attention on the last/new pair of components

$$\begin{cases} F^5 = 0 \\ G^5 = c(-5A^{100}+2B^{010}+C^{001}) = 0 \text{ automatically.} \end{cases}$$

Hence not possible to $\frac{\text{move away from}}{\text{move off}}$ p_c by the symmetries embeddable in flows.

But there may also exist symmetries not embeddable into flows! An independent, finite argument is needed.

So we have the point p_c, the values of the i.s.'s at p_c, but how does the entire distribution look like around $p_c \in 1.2.1.2.1$?

to the variable $x5$. In the down-to-earth practice it means that we are to work with the following 1-parameter subfamily of EKR's **1.2.1.2.1**:

$$\begin{aligned} dx0 - x1dt &= 0 & dy0 - y1dt &= 0\\ dt - x2\,dx1 &= 0 & dy1 - y2\,dx1 &= 0\\ dx2 - (1+x3)\,dx1 &= 0 & dy2 - y3\,dx1 &= 0\\ dx1 - x4\,dx3 &= 0 & dy3 - (1+y4)\,dx3 &= 0\\ dx4 - x5\,dx3 &= 0 & dy4 - (c+y5)\,dx3 &= 0\,. \end{aligned} \tag{30}$$

Geometrically this is a second order singularity: the mentioned tangency condition adds one codimension to the codimension two of the class 1.2.1.2.1: $2+1=3$.
Here $c \in \mathbb{R}$ is an arbitrary real parameter and these objects are understood as germs at $0 \in \mathbb{R}^{13}(t,\, x0,\, y0,\, x1,\, y1, \dots,\, x5,\, y5)$. We want to show that for every two different values c and $\widetilde{c}$ the distribution germs (30) are non-equivalent. To show this we start to analyze an arbitrary local diffeomorphism

$$\Phi \;=\; \big(T,\, X0,\, Y0,\, X1,\, Y1, \dots,\, X5,\, Y5\big) :\; \big(\mathbb{R}^{13},\, 0\big) \hookleftarrow$$

assumed to conjugate these two distributions. The general limitations on the function components of such a Φ are exactly as in the two previous sections. As to the additional limitations, in the discussed situation only the 2nd and 4th letters in the code 1.2.1.2.1 are not '1' – the class sits inside the sandwich class 1.$\underline{2}$.1.$\underline{2}$.1. Hence it is known from the beginning about the components $X2$ and $X4$ that

- $X2(t,\, x0,\, y0,\, x1,\, y1,\, x2,\, y2) = x2\, H(t,\, x0,\, y0,\, x1,\, y1,\, x2,\, y2)$,
- $X4(t,\, x0, \dots,\, y4) = x4\, G(t,\, x0, \dots,\, x4,\, y4)$

for certain invertible at 0 functions G, H. Moreover, identically as in the two previous sections, there must exist an invertible at 0 function f, $f\,|_0 \neq 0$, such that, altogether,

$$d\Phi(p)\begin{pmatrix} x4\begin{pmatrix} x2\begin{pmatrix}1\\ x1\\ y1\end{pmatrix}\\ 1\\ y2\\ 1+x3\\ y3\end{pmatrix}\\ 1\\ 1+y4\\ x5\\ c+y5\\ 0\\ 0\end{pmatrix} = f(p)\begin{pmatrix} x4\,G\begin{pmatrix} x2\,H\begin{pmatrix}1\\ X1\\ Y1\end{pmatrix}\\ 1\\ Y2\\ 1+X3\\ Y3\end{pmatrix}\\ 1\\ 1+Y4\\ X5\\ \widetilde{c}+Y5\\ *\\ *\end{pmatrix} \tag{31}$$

where $p = (t, x0, y0, \dots, x5, y5)$. In view of the first SEVEN components of the diffeomorphism Φ depending only on $t, x0, \dots, x2, y2$, the upper SEVEN among the scalar equations contained (or: hidden) in (31) can be divided sidewise by $x4$. While the upper THREE among them can be divided sidewise by the product of variables $x2\,x4$.
Agree, as in the preceding sections, to call thus simplified scalar equations 'level T', 'level $X0$', 'level $X4$', etc, in function of the row of $d\,\Phi(p)$ being involved. For instance, the level $Y0$ equation is the ∂_{y0}–component scalar equation in (31) divided sidewise by $x2\,x4$.

The relation binding the constants c and $\widetilde{c}$ is encoded in level $Y4$:

$$Y4_{x3} + Y4_{y3} + c\,Y4_{y4}\,|_0 = \widetilde{c}\,f\,|_0\,. \tag{32}$$

Observation 1.

$$Y4_{x3}\,|_0 = Y4_{y3}\,|_0 = 0\,.$$

Proof. The component function $Y4$ is expressed in terms of the function $Y3$ in level $Y3$:

$$x4(*) + Y3_{x3} + Y3_{y3}(1 + y4) = f(1 + Y4). \tag{33}$$

In (33) there also show up the functional coefficient f. It is explicitly got in level $X3$ in (31):

$$f = x4(*) + X3_{x3} + X3_{y3}(1 + y4). \tag{34}$$

Level $X1$ in (31) tells us that the function coefficient

$$fG \text{ depends only on } t,\, x0,\, y0, \dots,\, x2,\, y2. \tag{35}$$

In consequence, both the component functions $X3$ and $Y3$ showing up in levels $X2$ and $Y2$, respectively, are *affine* in the variable $x3$. Knowing this, we differentiate sidewise with respect to $x3$ at 0 the relations (33) and (34):

$$f\,Y4_{x3}\,|_0 = Y3_{x3x3} + Y3_{y3x3} - f_{x3}\,|_0 = Y3_{y3x3} - f_{x3}\,|_0,$$

$$f_{x3}\,|_0 = X3_{x3x3} + X3_{y3x3}\,|_0 = X3_{y3x3}\,|_0\,.$$

These relations yield together

$$f\,Y4_{x3}\,|_0 = Y3_{y3x3} - X3_{y3x3}\,|_0\,. \tag{36}$$

By the same reason as above the functions $X3$ and $Y3$ are affine in $y3$, and differentiating now the relations (33) and (34) sidewise with respect to $y3$ at 0,

$$f\,Y4_{y3}\,|_0 = Y3_{x3y3} + Y3_{y3y3} - f_{y3}\,|_0 = Y3_{x3y3} - f_{y3}\,|_0,$$

$$f_{y3}\,|_0 = X3_{x3y3} + X3_{y3y3}\,|_0 = X3_{x3y3}\,|_0,$$

which relations together imply

$$f\,Y4_{y3}\,|_0 = Y3_{x3y3} - X3_{x3y3}\,|_0\,. \tag{37}$$

Yet the functions: $X3$ (available in level $X2$) and $Y3$ (available in level $Y2$) do not have second order terms $x3\,y3$, neither. So the quantities on the right hand sides in (36) and (37) vanish. Observation 1 is proved.

In view of Observation 1 the key relation (32) reduces to

$$cY4_{y4}\,|_0 \;=\; \tilde{c}f\,|_0\,. \tag{38}$$

Lemma. $X3_{y3}\,|_0 \;=\; 0.$

Proof. The equation at level $X2$ reads in explicit terms

$$x2(\cdots) + \big(H + x2\,H_{x2}\big)\big(1 + x3\big) + x2\,y3\,H_{y2} \;=\; fG(1 + X3)$$

Differentiating this equation sidewise with respect to $y3$ at 0 and remembering that neither H nor fG depends on $y3$ (cf. (35)), $0 = fG\,X3_{y3}\,|_0$. Lemma is proved.

We are now in a position to replace $Y4$ in (38) by $Y3$. Namely, differentiating the relation (33) sidewise with respect to $y4$ at 0,

$$Y3_{y3}\,|_0 = \;f_{y4} + f\,Y4_{y4}\,|_0 = X3_{y3} + f\,Y4\,|_0 = \;f\,Y4\,|_0$$

by Lemma. Whence the key relation (38) reduces to

$$cY3_{y3}\,|_0 \;=\; \tilde{c}f^2\,|_0\,. \tag{39}$$

Observation 2. $f\,|_0 = 1.$

Proof. From (34) and Lemma there is

$$f\,|_0 = \;X3_{x3} + X3_{y3}\,|_0 = \;X3_{x3}\,|_0\;.$$

Let us write, for the reason of legibility only, $f\,|_0 =\; f_0$, $fG\,|_0 = \;(fG)_0$, $H\,|_0 = \;H_0$. Then, in the Taylor expansion of the function $X3$, $X3 = f_0\,x3 + \cdots$, and also

$$fG = (fG)_0 + \text{h.o.t.}\,, \qquad\qquad H = H_0 + \text{h.o.t.} \tag{40}$$

where in both the higher order terms above there is no coordinate $x3$ whatsoever (cf. also (35)). Let us write in explicit terms the equation in level $X2$

$$x2(\cdots) + \Big(H_0 + \text{h.o.t.} + x2\,H_{x2}\Big)\big(1 + x3\big) + x2(\cdots) = \Big((fG)_0 + \text{h.o.t.}\Big)\big(1 + f_0\,x3 + \cdots\big). \tag{41}$$

Now, by comparing the zero order terms in (41)

$$H_0 = (fG)_0\,, \tag{42}$$

while comparing the $(x3)^1$ terms in (41) yields

$$H_0 = (fG)_0 \cdot f_0\,. \tag{43}$$

The relations (42) and (43) together yield $f_0 = 1$. Observation 2 is proved.

The relation (33) enhanced by Observation 2 clearly implies

$$Y3_{x3} + Y3_{y3}\,|_0 = 1. \tag{44}$$

Remark. At this moment what only remains to show is that the first summand $Y3_{x3}\,|_0$ in (44) vanishes. This follows from a rather compact series of inferences.

Firstly, level $Y1$ written in explicit terms

$$fG \cdot Y2 = x2\,(*) + Y1_{x1} + y2\,(*)$$

implies that

$$0 = fG \cdot Y2\,|_0 = Y1_{x1}\,|_0 \tag{45}$$

and

$$fG \cdot Y2_{x1}\,|_0 = Y1_{x1x1}\,|_0\,. \tag{46}$$

Secondly, level $Y0$ written in explicit terms

$$fGH \cdot Y1 = Y0_t + x1\,Y0_{x0} + y1\,Y0_{y0}$$

implies

$$0 = (fGH{\cdot}Y1)_{x1x1}\,|_0 = 0{+}2(fGH)_{x1}{\cdot}Y1_{x1}{+}fGH{\cdot}Y1_{x1x1}\,|_0 = fGH{\cdot}Y1_{x1x1}\,|_0$$

by (45). Hence

$$Y2_{x1}\,|_0 = 0 \tag{47}$$

by (46). Thirdly, level $Y2$ written in explicit terms

$$fG \cdot Y3 = x2\,(*) + Y2_{x1} + y2\,(*) + (1 + x3)Y2_{x2} + y3\,Y2_{y2}$$

implies that

$$0 = Y2_{x1} + Y2_{x2}\,|_0 \tag{48}$$

and also that

$$fG \cdot Y3_{x3}\,|_0 = Y2_{x2}\,|_0\,. \tag{49}$$

Now (47) and (48) yield $Y2_{x2}\,|_0 = 0$. This latter equality coupled with (49) implies $Y3_{x3}\,|_0 = 0$.
As it has been already noticed, this information ends the entire proof, implying by (44) that $Y3_{y3}\,|_0 = 1$. Hence, all in all, giving $c \cdot 1 = \widetilde{c} \cdot 1$ (compare (39) and Observation 2).

Every two *different* values c and $\widetilde{c}$ of the parameter in the (pseudo) normal form (30) are non-equivalent.

Proofs for the classes 1.2.2.1.2 and 1.2.3.1.2 are different, but shorter. How about the remaining 38 singularity classes in length 5? $\left(\frac{1}{2}(3^{5-1}+1)-3=38\right)$

For instance 1.2.1.3.1 whose father is 1.2.1.3 – known to be the union of three orbits. The germs in 1.2.1.3.1 are (up to equivalence) one-step prolongations of chosen representatives of those orbits.

Of particular interest was the second orbit inside 1.2.1.3 (of relative codimension 1), because it brought Weber's implementation to a momentary stop.

The prolongation of its model representative is

$$dx0 - x1\,dt = 0 = dy0 - y1\,dt$$
$$dt - x2\,dx1 = 0 = dy1 - y2\,dx1$$
$$dx2 - x3\,dx1 = 0 = dy2 - (1+y3)\,dx1$$
$$dx1 - x4\,dy3 = 0 = dx3 - y4\,dy3$$
$$dx4 - (b+x5)\,dy3 = 0 = dy4 - (c+y5)\,dy3$$

After freezing to 0 the components of the i.s.'s at levels 0, 1, 2, 3, 4 and after removing a bug in the implementation, the expression for the i.s.'s coming from the functions A, B, C (now at $0 \in \mathbb{R}^{13}(t, x0, y0, \dots, x5, y5)$) is

$$\left(-B_{x0} + C_{y0}\right) b\, \partial_{x5} + \left(4B_{x0} - 3C_{y0}\right) c\, \partial_{y5}\Big|_0$$

For $b \neq 0 \neq c$ these can vary independently one from the other. Hence potentially emerging moduli are excluded.