Translation length formula for two-generated groups acting on trees

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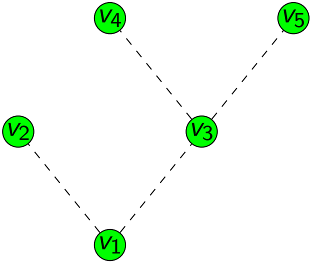
What is a tree?

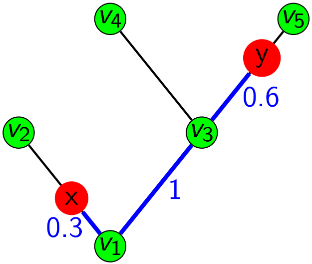
Combinatorial tree $T = (V(T), E(T))$



$$V(T) = \{v_1, \ldots, v_5\}$$
$$E(T) = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_3, v_4\}, \{v_3, v_5\}\}$$
$$d \colon V(T) \times V(T) \to \mathbb{Z}$$
e.g. $d(v_2, v_5) = 3$

What is a tree?

The geometric realization (X, d) of T



$$X = \left(\bigsqcup_{e \in E(T)} [0,1]_{\mathbb{R}} \right) \Big/ \sim$$

$d \colon X \times X \to \mathbb{R}$

e.g. $d(x, y) = 1.9$

$\mathbb{R}$-tree

A *segment* joining $x, y \in X$ in a metric space (X, d) is the image of an isometric embedding $i\colon [0, d(x,y)]_{\mathbb{R}} \to X$, $i(0) = x$, $i(d(x,y)) = y$.

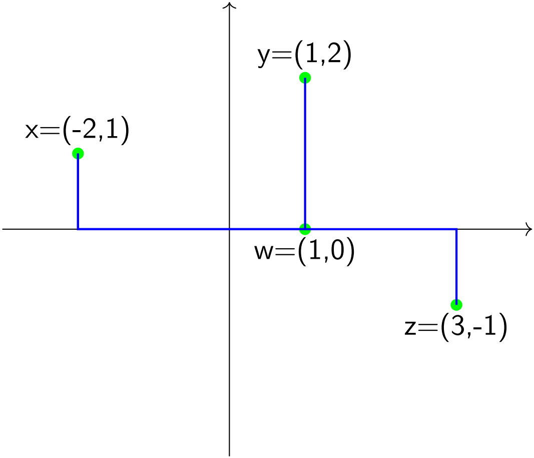
Definition

An $\mathbb{R}$-*tree* is a metric space (X, d) such that:

1. any $x, y \in X$ can be joined by a unique segment, denoted by $[x, y]$;
2. $[x, y] \cap [x, z]$ is a segment with x as one endpoint;
3. $[x, y] \cup [x, z] = [y, z]$ if $[x, y] \cap [x, z] = \{x\}$.

An $\mathbb{R}$-tree that is not the realization of any combinatorial tree

$X = \mathbb{R}^2$, d – "jungle river" metric



$d(x, y) = 6,\ d(x, z) = 7$
$[x, y] \cap [x, z] = [x, w]$
$[w, y] \cup [w, z] = [y, z]$

Λ-metric spaces

$(\Lambda, +, \leq)$ – a nontrivial totally ordered Abelian group
Examples: $\Lambda \leq \mathbb{R}$, $\mathbb{Z} \oplus \mathbb{Z}$, $\mathbb{R} \oplus \mathbb{R}$ (with the lexicographic order), ${}^*\mathbb{R}$ (hyperreal numbers)

- We can define a Λ-*metric* $d\colon X \times X \to \Lambda$ on X and a Λ-*metric space* (X, d).
- The family of open balls in (X, d) is a base of a normal Hausdorff topology on X.
- We naturally define an isometry between Λ-metric spaces.
- The group Λ is a Λ-metric space with the Λ-metric $d(x, y) := |x - y|$, where $|\lambda| := \max\{\lambda, -\lambda\}$.
- An interval in Λ is defined as $[a, b]_\Lambda := \{\lambda \in \Lambda\colon a \leq \lambda \leq b\}$ for $a \leq b$.

Λ-trees

The definition of an $\mathbb{R}$-tree can be generalized to Λ.

Definition

A Λ-*tree* is a Λ-metric space (X, d) such that:

1. any $x, y \in X$ can be joined by a unique segment, denoted by $[x, y]$;
2. $[x, y] \cap [x, z]$ is a segment with x as one endpoint;
3. $[x, y] \cup [x, z] = [y, z]$ if $[x, y] \cap [x, z] = \{x\}$.

Remark

$\mathbb{Z}$-trees correspond exactly to combinatorial trees.

Isometries of Λ-trees

An isometry $g\colon X \to X$ of a Λ-tree X can be one of three types:

1. **elliptic isometry** – g fixes a point in X;
2. **inversion** – g has no fixed point, but g^2 does;
3. **hyperbolic isometry** – g^2 has no fixed point.

Remark

If $\Lambda = 2\Lambda$ (e.g., $\Lambda = \mathbb{R}$), there are no inversions.

Translation length and axis of an isometry

Definition

The *translation length* of an isometry g of a Λ-tree (X, d) is

$$\|g\| := \begin{cases} 0, & \text{if } g \text{ is an inversion,} \\ \min\{d(x, gx) \colon x \in X\} & \text{otherwise.} \end{cases}$$

Definition

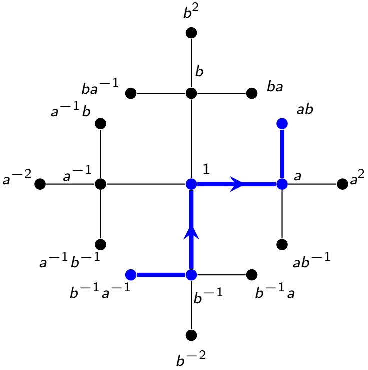
The *axis* of a hyberbolic isometry $g \colon X \to X$ is the set

$$A_g := \{x \in X \colon d(x, gx) = \|g\|\}.$$

It is a nonempty subtree of X isometric to a convex subset of Λ; the action of g on A_g corresponds to the translation by $\|g\| > 0$.

Example 1 (Cayley graph of the free group $F(a, b)$)

We treat $X = F(a, b)$ as a $\mathbb{Z}$-tree; $F(a, b)$ acts on X by left multiplication.



$\|ab\| = 2$

$A_{ab} = \{\ldots, b^{-1}a^{-1}, b^{-1}, 1, a, ab, \ldots\}$

Example 2 (Bruhat–Tits tree)

Let K be a field with a *non-Archimedean valuation* v, i.e., a homomorphism $v\colon K^* \to \Lambda$ such that $v(a+b) \geq \min\{v(a), v(b)\}$ for $a+b \neq 0$.

There exists a Λ-tree X_v and an isometric action of $\mathrm{GL}(2,K)$ on X_v with the translation length

$$\|g\| = \max\{v(\det g) - 2v(\operatorname{tr} g), 0\}.$$

Remarks

1. The action of $\mathrm{SL}(2,K)$ on X_v is without inversions.
2. If $\Lambda = \mathbb{Z}$, X_v is an infinite, regular combinatorial tree.

Pseudo-length

Definition

A function $\|\cdot\|: G \to \Lambda_+$ is called a *pseudo-length* if, for all $g, h \in G$, it satisfies the conditions:

1. $\max\{0, \|gh\| - \|g\| - \|h\|\} \in 2\Lambda$ if $\|g\| > 0$, $\|h\| > 0$;
2. $\|ghg^{-1}\| = \|h\|$;
3. $\|gh\| = \|gh^{-1}\|$ or $\max\{\|gh\|, \|gh^{-1}\|\} \leq \|g\| + \|h\|$;
4. $\|gh\| = \|gh^{-1}\| > \|g\| + \|h\|$ or $\max\{\|gh\|, \|gh^{-1}\|\} = \|g\| + \|h\|$ if $\|g\| > 0$, $\|h\| > 0$.

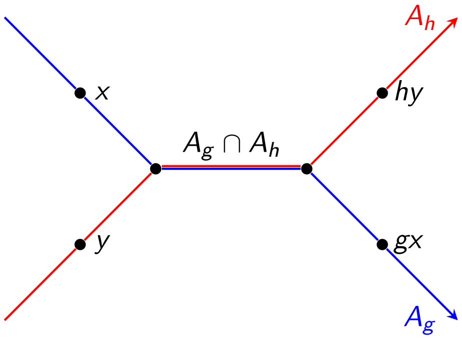
Theorem (Parry 1988)

A function $\|\cdot\|: G \to \Lambda_+$ is a pseudo-length if and only if there exists a Λ-tree X and an action of G on X by isometries such that $\|g\|$ is the translation length of g for $g \in G$.

Ping-pong pair

Definition

We call $(g, h) \in G \times G$ a *ping-pong pair* if $\|g\| > 0$, $\|h\| > 0$, and $|\|g\| - \|h\|| < \min\{\|gh\|, \|gh^{-1}\|\}$.
Geometrically, it means that $A_g \cap A_h = \emptyset$ or $A_g \cap A_h$ is a segment "shorter" than $\min\{\|g\|, \|h\|\}$.



Properly discontinuous actions

An action of G on a topological space X is *properly discontinuous* if, for each $x \in X$, there exist a neighborhood $U \ni x$ such that $g(U) \cap U = \emptyset$ for $g \neq 1_G$.

Theorem (Culler–Morgan 1985, Chiswell 1994)

If G acts on a Λ-tree X by isometries and $(a, b) \in G \times G$ is a ping-pong pair, then the subgroup $\langle a, b \rangle$ is **free of rank two***, and acts on X* **properly discontinuously** *and* **without inversions***.*

Proof methods: referring to the geometry of Λ-trees, "ping-pong lemma".

Explicit translation length formula

Theorem (Orzechowski 2025)

If $\|\cdot\|$ is a pseudo-length on G and (a, b) is a ping-pong pair, then

$$2\|w\| = \left(\sum_{i=1}^{n-1} \|x_i x_{i+1}\|\right) + \|x_n x_1\| > 0.$$

for any cyclically reduced word $w = x_1 \ldots x_n$, $n \geq 1$, $x_i \in \{a, b, a^{-1}, b^{-1}\}$.

Proof methods: a combinatorial approach using the axioms of a pseudo-length, induction on word length.

Another form of the formula

Let $w = a^{m_1} b^{n_1} \dots a^{m_k} b^{n_k}$, $k \geq 1$, and $m_1, \dots, m_k, n_1, \dots, n_k \in \mathbb{Z} \setminus \{0\}$. Then

$$\|w\| = \|a\| \sum_{i=1}^{k} (|m_i| - 1) + \|b\| \sum_{i=1}^{k} (|n_i| - 1) + \frac{N}{2} \|ab^{-1}\| + \frac{2k - N}{2} \|ab\|,$$

where N denotes the number of sign changes in the sequence $(m_1, n_1, \dots, m_k, n_k, m_1)$.

Corollary

The group $\langle a, b \rangle$ is free of rank two, and acts properly discontinuously and without inversions on the corresponding Λ-tree X.

Inverse problem: construction of a pseudo-length

Let $\alpha, \beta, \gamma, \delta \in \Lambda$ satisfy the conditions

$$\begin{aligned} &\gamma - \alpha - \beta \in 2\Lambda, \quad \delta - \alpha - \beta \in 2\Lambda; \\ &\gamma = \delta > \alpha + \beta \quad \text{or} \ \max\{\gamma, \delta\} = \alpha + \beta; \\ &\alpha > 0, \ \beta > 0, \ |\alpha - \beta| < \min\{\gamma, \delta\}. \end{aligned} \tag{1}$$

Let $\Sigma := \{a, b, a^{-1}, b^{-1}\}$ and define $f\colon \Sigma \times \Sigma \to \Lambda$ as follows:
$f(a,a) = 2\alpha$, $f(b,b) = 2\beta$, $f(a,b) = \gamma$, $f(a,b^{-1}) = \delta$,
$f(x,y) = f(y,x) = f(y^{-1},x^{-1})$, $f(x,x^{-1}) = 0$ for $x, y \in \Sigma$.

Theorem (Orzechowski 2025)

Let $\|1\| := 0$, *and for* $w \neq 1$ *put*

$$\|w\| := \frac{1}{2}\left(\sum_{i=1}^{n-1} f(x_i, x_{i+1}) + f(x_n, x_1)\right),$$

where $x_1 \dots x_n$ *is a cyclically reduced word conjugate to* w. **Then** $\|\cdot\|$ **is a pseudo-length on** $F(a,b)$.

Conlusion: existence and uniqueness

Corollary

If $\alpha, \beta, \gamma, \delta \in \Lambda$ *satisfy the conditions* (1), *then there exists a unique pseudo-length* $\|\cdot\| \colon F(a,b) \to \Lambda_+$ *such that* $\|a\| = \alpha$, $\|b\| = \beta$, $\|ab\| = \gamma$, $\|ab^{-1}\| = \delta$.

We denote this pseudo-length by $\|\cdot\|_{\alpha,\beta,\gamma,\delta}$.

Application: purely hyperbolic pseudo-lengths

Definition

A pseudo-length $\|\cdot\|\colon G \to \Lambda_+$ is called *purely hyperbolic* if $\|g\| > 0$ for all $g \neq 1_G$. It corresponds to the translation length function of a **free** action **without inversions** on a Λ-tree.

Theorem

Let $\{0\} \neq \Lambda \leq \mathbb{R}$ and $\|\cdot\|\colon F(a,b) \to \Lambda_+$ be a purely hyperbolic pseudo-length. There exists an automorphism σ of $F(a,b)$ and $\alpha, \beta, \gamma, \delta \in \Lambda$ satisfying (1) such that

$$\|w\| = \|\sigma(w)\|_{\alpha,\beta,\gamma,\delta} \quad \textit{for } w \in F(a,b).$$

Finding the automorphism σ involves performing a finite sequence of Nielsen transformations on the basis (a,b) until we get a basis (g,h) that is a ping-pong pair.

Algorithm (Culler–Vogtmann 1988, Chiswell 1994)

Input: a purely hyperbolic pseudo-length $\|\cdot\|: F(a,b) \to \mathbb{R}_+$
Output: a ping-pong pair generating $F(a,b)$

```
(g, h) := (a, b)
if ‖g‖ < ‖h‖ then
    (g, h) := (h, g)
if ‖gh‖ < ‖gh⁻¹‖ then
    (g, h) := (g, h⁻¹)
while ‖g‖ − ‖h‖ = ‖gh⁻¹‖ do
    (g, h) := (gh⁻¹, h)
    if ‖g‖ < ‖h‖ then
        (g, h) := (h, g)
    if ‖gh‖ < ‖gh⁻¹‖ then
        (g, h) := (g, h⁻¹)
if ‖g‖ − ‖h‖ < ‖gh⁻¹‖ then
    return (g, h)
else
    return (gh⁻¹, h)
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