$\mathcal{C}^\infty$-structures approach for the travelling wave solutions of the gKdV equation

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Introduction to gKdV Equation

Korteweg−de Vries (KdV) Equation:

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- Describes wave evolution in shallow water.

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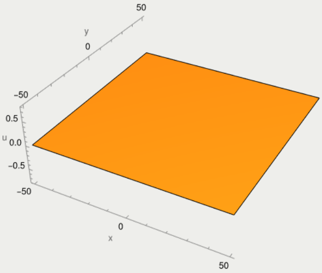
Generalized KdV (gKdV) Equation:

$$u_t + u_{xxx} + a(u)u_x = 0, \tag{2}$$

where $a \in C^\infty(\mathbb{R})$.

Travelling Wave Solutions



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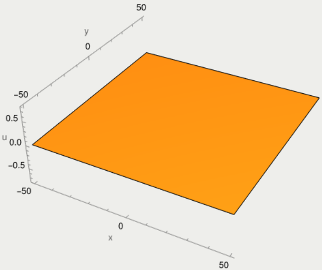

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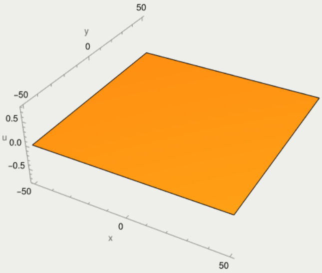

- Apply transformation
  $z = x - ct, \quad y(z) = u(x, t).$
- Leads to ODE:

$$u_t + u_{xxx} + a(u)u_x = 0$$
$$\downarrow$$
$$\boxed{-cy' + y''' + a(y)y' = 0}$$

$\mathcal{C}^\infty$-structures integration method

See **(Pan-Collantes et al., 2023c, Pan-Collantes et al., 2023b, Pan-Collantes et al., 2023a, Pan-Collantes et al., 2025)**

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Associated Distribution

The associated distribution to the ODE is generated by:

$$Z = \partial_z + y_1\partial_y + y_2\partial_{y_1} + (c - a(y))y_1\partial_{y_2}.$$

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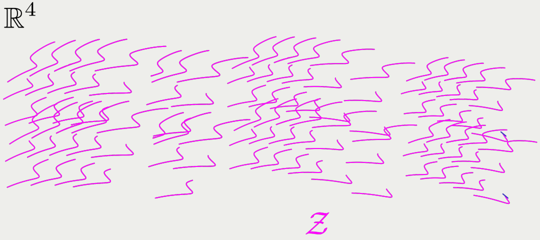

$$z \mapsto (z, f(z), f'(z), f''(z))$$

The integral manifolds of this distribution correspond to the prolongation of solutions of the ODE.

$\mathcal{C}^\infty$-structure for an ODE

Definition ($\mathcal{C}^\infty$-structure)

An ordered collection of vector fields $\langle X_1, \ldots, X_m \rangle$ is a $\mathcal{C}^\infty$**-structure** for an ODE with associated vector field Z if the distribution

$$\mathcal{S}(\{Z, X_1, \ldots, X_i\})$$

has constant rank $i + 1$ and is involutive for each i such that $1 \leq i \leq m$.

Generalization of solvable structures: **(Basarab-Horwath, 1991)**.

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Involutivity

Given $X, Y \in \mathcal{S}(\{Z, X_1, \ldots, X_i\})$, we have $[X, Y] \in \mathcal{S}(\{Z, X_1, \ldots, X_i\})$.

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Frobenius Theorem

A distribution is involutive if and only if it admits a local integral manifold.

$\mathcal{C}^\infty$-structure

In $\mathbb{R}^4$ with coordinates (z, y, y_1, y_2), we have the following vector field:

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Involutive Distributions

- $\mathcal{S}(\{Z\})$, rank 1
- $\mathcal{S}(\{Z, X_1\})$, rank 2
- $\mathcal{S}(\{Z, X_1, X_2\})$, rank 3
- $\mathcal{S}(\{Z, X_1, X_2, X_3\})$, rank 4

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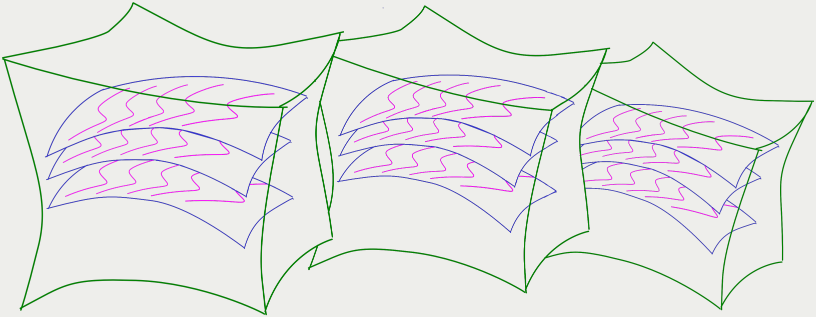
$\mathcal{C}^\infty$-structure: visually

$\mathbb{R}^4$



Integration Method using $\mathcal{C}^\infty$-structures

Given a $\mathcal{C}^\infty$-structure $\langle X_1, X_2, X_3 \rangle$ for the ODE Z:

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1. **Define 1-forms:** Volume form $\mathbf{\Omega} = dz \wedge dy \wedge dy_1 \wedge dy_2$.

$$\omega_3 = X_2 \lrcorner X_1 \lrcorner Z \lrcorner \mathbf{\Omega}$$

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2. **Pfaffian Equation:**

$$\boxed{\omega_3 \equiv 0}$$

is completely integrable.
Find a first integral $I_3 = I_3(z, y, y_1, y_2)$ by solving the linear homogeneous PDE

$$dI_3 \wedge \omega_3 = 0.$$

Integration Method using $\mathcal{C}^\infty$-structures

3. **Define Level Sets:** Consider level sets $I_3 = C_3$ ($C_3 \in \mathbb{R}$). Find a local parametrization ι_3.

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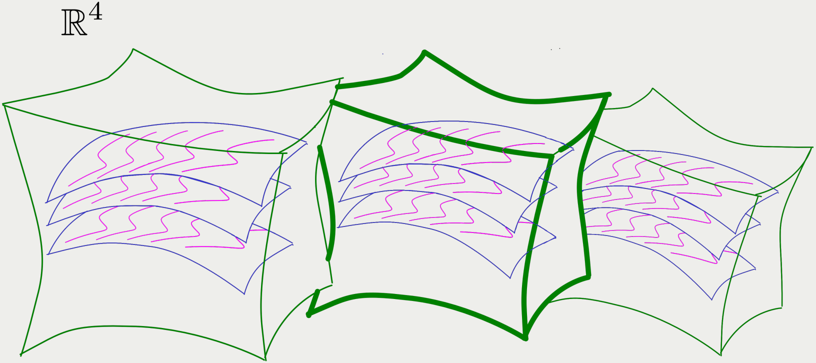

$$I_3(z, y, y_1, y_2) = C_3$$

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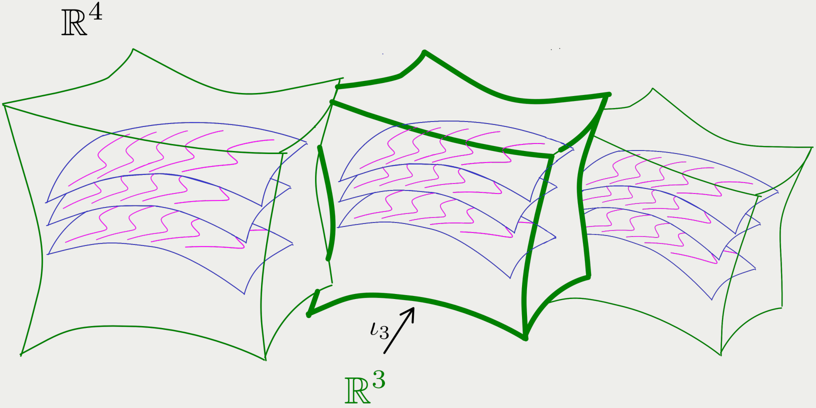

Integration Method using $\mathcal{C}^\infty$-structures

4. **Restrict Forms and Repeat:** Pullback

$$\omega_2|_{\{I_3=C_3\}} := \iota_3^*(\omega_2)$$
$$\omega_1|_{\{I_3=C_3\}} := \iota_3^*(\omega_1)$$

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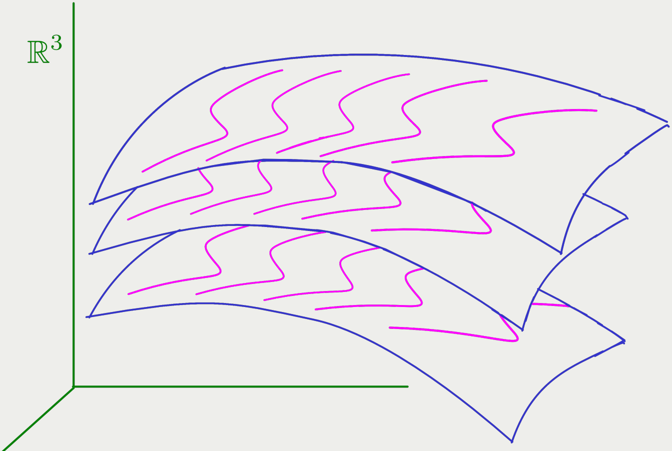

Integration Method using $\mathcal{C}^\infty$-structures

5. **Compose the parametrizations**: Composition

$$\iota_3 \circ \iota_2 \circ \iota_1 : \mathbb{R} \to \mathbb{R}^4$$

is a local parametrization of the integral curves of Z.

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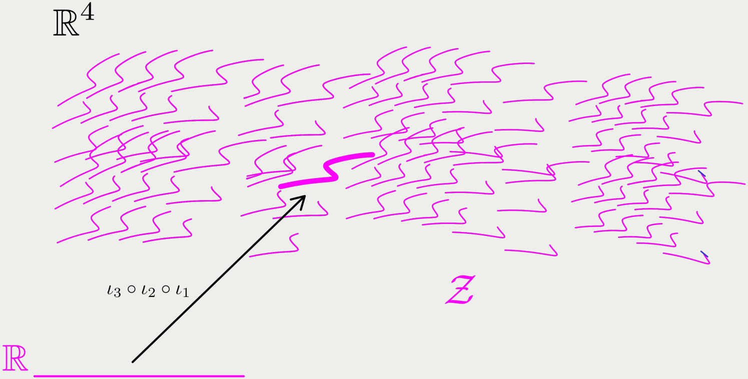

Integration Method using $\mathcal{C}^{\infty}$-structures

Solutions

- Parametric expression: $\iota_3 \circ \iota_2 \circ \iota_1$

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- Parametric expression: $\iota_3 \circ \iota_2 \circ \iota_1$
- Explicit expression: the parametrization is of the form $(z, f(z), f'(z), f''(z))$.
- Implicit expression:

$$I_1(z, y; C_2, C_3) = C_1.$$

The gKdV ordinary differential equation

Third-Order ODE Vector Field

$$Z = \partial_z + y_1\partial_y + y_2\partial_{y_1} + (cy_1 - a(y)y_1)\partial_{y_2}$$

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$\mathcal{C}^\infty$-structure needed!

First vector field

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First vector field in the $\mathcal{C}^\infty$-structure

$$X_1 = \partial_z$$

Ansatz for X_2

Vector Field Ansatz

$$X_2 = \partial_{y_1} + \eta(z, y, y_1)\partial_{y_2}$$

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Involutivity Condition

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Determining Equations

Conditions

$$\eta^2 y_1 + \eta_y y_1^2 + \eta_{y_1} y_1 y_2 - \eta y_2 + \eta_z y_1 = 0$$
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Particular Solution

$$\eta = \frac{y_1}{y}$$

$\mathcal{C}^\infty$-structure

Third vector field

$$X_3 = \partial_{y_2}$$

Third vector field

$$X_3 = \partial_{y_2}$$

$\mathcal{C}^\infty$-structure for $Z = \partial_z + y_1\partial_y + y_2\partial_{y_1} + (cy_1 - a(y)y_1)\partial_{y_2}$

$$X_1 = \partial_z$$
$$X_2 = \partial_{y_1} + \frac{y_1}{y}\partial_{y_2}$$
$$X_3 = \partial_{y_2}$$

$\mathcal{C}^\infty$-structure-based method

1-Forms

$$\omega_1 = -y_1\, dz + dy$$
$$\omega_2 = -y_2\, dy + y_1\, dy_1$$
$$\omega_3 = -y_1 \left(\frac{y_2}{y} + a(y) - c \right) dy + \frac{y_1^2}{y} dy_1 - y_1 dy_2$$

$\mathcal{C}^\infty$-structure-based method

1-Forms

$$\omega_1 = -y_1\, dz + dy$$

$$\omega_2 = -y_2\, dy + y_1\, dy_1$$

$$\omega_3 = \boxed{-y_1 \left(\frac{y_2}{y} + a(y) - c \right) dy + \frac{y_1^2}{y} dy_1 - y_1 dy_2 \equiv 0}$$

Solution of ω_3

First Integral

$$I_3 = yy_2 - \frac{1}{2}y_1^2 - H_1(y)$$

where

$$H_1(y) = \int y(c - a(y))dy$$

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Level Set

$$\{(z, y, y_1, y_2) \in \mathbb{R}^4 : I_3(z, y, y_1, y_2) = C_3\}$$

Restriction to the level set

Parametrization

$$\iota_3(z, y, y_1) = \left(z, y, y_1, \frac{y_1^2 + 2H_1(y) + 2C_3}{2y}\right)$$

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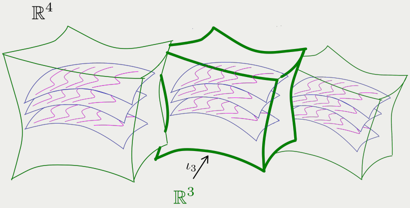

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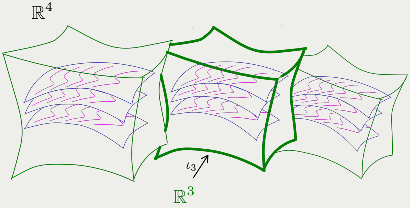

$$\omega_3\big|_{\{I_3 = C_3\}} = 0$$

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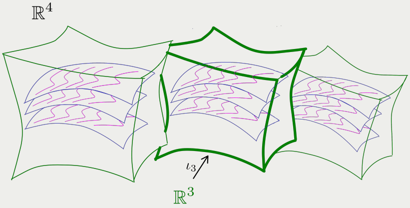

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Second Pfaffian equation

Solution to $\omega_2\big|_{I_3=C_3} \equiv 0$

$$I_2 = \frac{y_1^2 - 2yH_2(y) + 2C_3}{y}$$

where

$$H_2(y) = \int \frac{H_1(y)}{y^2} dy$$

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Level Set

$$\{(z, y, y_1) \in \mathbb{R}^3 : I_2(z, y, y_1; C_3) = C_2\}$$

Restriction to the level set

Parametrization

$$\iota_2 : (z, y) \to \left(z, y, \sqrt{y(C_2 + 2H_2(y)) - 2C_3}\right)$$

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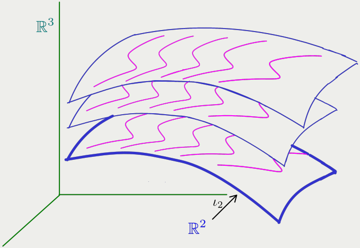

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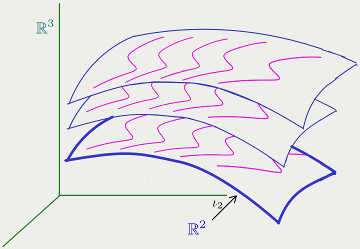

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Last iteration

Solution to $\omega_1\big|_{\{I_2=C_2\}} \equiv 0$

$$I_1 = z - H_3(y; C_2, C_3), \quad H_3(y) = \int \frac{1}{\sqrt{y(C_2 + 2H_2(y)) - 2C_3}} dy$$

General Solution

Implicit Solution

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Traveling Wave

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Explicit expressions

The functions H_1, H_2, H_3 and F depend on the function a.

Particular cases of the gKdV equation

KdV equation: $a(u) = 6u$

$$u(x,t) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(-x + ct + C_1)\right) \quad \text{for } c > 0$$

$$u(x,t) = \frac{c}{2}\sec^2\left(\frac{\sqrt{-c}}{2}(-x + ct + C_1)\right) \quad \text{for } c < 0$$

Modified KdV: $a(u) = u^2$

$$u(x,t) = \frac{144c\, e^{\sqrt{c}(C_1 - x + ct)}}{e^{2\sqrt{c}(C_1 - x + ct)} + 864c} \quad \text{for } c > 0$$

Generalized KdV: $a(u) = \frac{u^n}{n}$

$$u(x,t) = \left(\frac{cn(n+1)(n+2)}{2\cosh^2\left(\frac{n}{2}\sqrt{c}(C_1 - x + ct)\right)} \right)^{\frac{1}{n}} \quad \text{for } c > 0$$

$$u(x,t) = \left(\frac{cn(n+1)(n+2)}{2\cos^2\left(\frac{n}{2}\sqrt{c}(C_1 - x + ct)\right)} \right)^{\frac{1}{n}} \quad \text{for } c < 0$$

Schamel–KdV: $a(u) = \alpha\sqrt{u} + \beta u$

For $c > 0$:

$$u(x,t) = \left(\frac{900c\, e^{\frac{1}{2}\sqrt{c}(C_1 - x + ct)}}{240\alpha\, e^{\frac{1}{2}\sqrt{c}(C_1 - x + ct)} + e^{\sqrt{c}(C_1 - x + ct)} + 67500\beta c + 14400\alpha^2} \right)^2$$

For $c < 0$:

$$u(x,t) = \frac{225c^2 \left(8\alpha\sqrt{\xi}\sin\left(\frac{\sqrt{-c}(C_1 - x + ct)}{2}\right) + 75\beta c - \xi\cos^2\left(\frac{\sqrt{-c}(C_1 - x + ct)}{2}\right)\right)}{\left(75\beta c - \xi\cos^2\left(\frac{\sqrt{-c}(C_1 - x + ct)}{2}\right)\right)^2}$$

where $\xi = 16\alpha^2 + 75\beta c$.

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- **Geometric method:** Use of C^∞*-structures*, a generalization of solvable structures, to integrate the ODE derived from the travelling wave reduction.
- **General idea:** Flag of foliations.
- **Result:** A unified and general method to derive explicit solutions for a broad family of gKdV-type equations:

$$u(x,t) = F(x - ct - C_1; C_2, C_3)$$

.

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Thanks!

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