On the asymptotic behavior at infinity of solutions of the Beltrami equation with two characteristics

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Beltrami equation with two characteristics

Let D be a domain in the complex plane $\mathbb{C}$, i.e., a connected and open subset of $\mathbb{C}$, and let μ and $\nu\colon D \to \mathbb{C}$ be a measurable functions with $|\mu(z)| + |\nu(z)| < 1$ a.e. (almost everywhere) in D. We study the *Beltrami equation with two characteristics*

(1) $$f_{\bar{z}} = \mu(z) f_z + \nu(z) \overline{f_z} \quad \text{a.e.} \quad \text{in} \quad D,$$

where $f_{\bar{z}} = (f_x + i f_y)/2$, $f_z = (f_x - i f_y)/2$, $z = x + iy$, f_x and f_y are the partial derivatives of f by x and y, respectively. The functions μ and ν are called the *complex coefficients* and

(2) $$K_{\mu,\nu}(z) = \frac{1 + |\mu(z)| + |\nu(z)|}{1 - |\mu(z)| - |\nu(z)|}$$

the *dilatation quotient* for the equation (1).
The Beltrami equation (1) is said to be *degenerate* if $\operatorname{ess\,sup} K_{\mu,\nu}(z) = \infty$.

Beltrami equation with two characteristics

Picking $\nu(z) \equiv 0$ in (1), we arrive at the standard *Beltrami equation* of the form

$$f_{\overline{z}} = \mu(z) f_z . \tag{3}$$

For the equation (3) we set

$$K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} . \tag{4}$$

The Beltrami equation (3) is said to be *degenerate* if $\operatorname{ess\,sup} K_\mu(z) = \infty$.

Beltrami equation with two characteristics

Theorems on the existence of homeomorphic solutions of the Sobolev class $W^{1,1}_{loc}$ have been recently proved by the method of moduli for many linear and quasilinear degenerate Beltrami equations; see, for example, [1], [2], [3], [4], [5], [6].

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[1] Astala, K., Iwaniec, T., Martin, G.J. (2009). *Elliptic Differential Equations and Quasiconformal Mappings in the Plane.* Princeton Mathematical Series, **48**, Princeton University Press, Princeton.

[2] Sevost'yanov, E. (2023). *Mappings with Direct and Inverse Poletsky Inequalities*. Developments in Mathematics, **78**. Springer, Cham.

[3] Gutlyanskii, V., Ryazanov, V., Srebro, U., Yakubov, E. (2012). *The Beltrami equations: A geometric approach. Developments in Math., 26.* Springer, New York.

[4] Martio, O., Ryazanov, V., Srebro, U., Yakubov, E. (2009). *Moduli in modern mapping theory. Springer Monographs in Mathematics.* Springer, New York.

[5] Dovhopiatyi, O., Sevost'yanov, E. (2022). On the existence of solutions of quasilinear Beltrami equations with two characteristics. *Ukrainian Mathematical Journal, 74*(7), 1099–1112.

[6] Ryazanov, V., Salimov, R., Sevost'yanov, E. (2023). On Hölder Continuity of Solutions to the Beltrami Equations. *Ukr. Math. J., 75*(4), 1099–1112.

Beltrami equation with two characteristics

Picking $\mu(z) \equiv 0$ in (1), we arrive at the *Beltrami equation of the second type*

(5) $$f_{\overline{z}} = \nu(z)\overline{f_z}.$$

For the equation (5) we set

(6) $$K_\nu(z) = \frac{1+|\nu(z)|}{1-|\nu(z)|}.$$

This equation plays a great role in many problems of mathematical physics, see e.g. [7].

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[7] S. L. Krushkal' and R. Kühnau, *Quasiconformal mappings, new methods and applications.* Nauka, Novosibirsk, 1984 [in Russian]; *Quasikonfoeme Abbildungen-neue Methoden und Anwendungen.* Teubner-Text zur Mathematik, **54**, BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1983 [in German].

GFMV functions

For a point $z_0 \in \mathbb{C}$ and $r > 0$, let us set

$$B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}.$$

We say that a function $\varphi\colon \mathbb{C} \to \mathbb{R}$ has a *global finite mean value* at the point $z_0 \in \mathbb{C}$, abbr. $\varphi \in \mathrm{GFMV}(z_0)$, if

$$\limsup_{R\to\infty} \frac{1}{\pi R^2} \int\limits_{B(z_0,R)} |\varphi(z)|\,dxdy < \infty. \tag{7}$$

Here condition (7) includes the assumption that φ is locally integrable in $\mathbb{C}$.

GFMV functions

Proposition 1. *If the function* $\varphi\colon \mathbb{C} \to \mathbb{R}$ *has a global finite mean value at the point* $z_0 \in \mathbb{C}$*, then* φ *has a global finite mean value at every point* $\zeta \in \mathbb{C}$*.*

We say that a function $\varphi\colon \mathbb{C} \to \mathbb{R}$ has a *global finite mean value* in $\mathbb{C}$, abbr. $\varphi \in \mathrm{GFMV}(\mathbb{C})$, if $\varphi \in \mathrm{GFMV}(z_0)$ for some point $z_0 \in \mathbb{C}$.

GFMV functions

Proposition 2. *Let* $z_0 \in \mathbb{C}$, $C > 0$ *and* $r_0 > 0$. *If a nonnegative function* $\varphi \colon \mathbb{C} \to \mathbb{R}$ *satisfies the condition*

$$\frac{1}{2\pi R} \int\limits_{|z-z_0|=R} \varphi(z)|dz| \leqslant C$$

for a.a. $R \in (r_0, +\infty)$, *then* φ *has a global finite mean value at the point* z_0.

Corollary 1. *Let* $z_0 \in \mathbb{C}$, $C > 0$ *and* $r_0 > 0$. *If a nonnegative function* $\varphi \colon \mathbb{C} \to \mathbb{R}$ *satisfies the condition* $\varphi(z) \leqslant C$ *for a.a.* $z \in \mathbb{C}$, *then* φ *has a global finite mean value at the point* z_0.

GFMV functions

Lemma 1. *Let* $z_0 \in \mathbb{C}$*. If a nonnegative function* $\varphi\colon \mathbb{C} \to \mathbb{R}$ *has a global finite mean value at* z_0*, then for* $R > e$

(8) $$\int_{\mathbb{A}(z_0,e,R)} \frac{\varphi(z)\,dxdy}{|z-z_0|^2} \leqslant C \log R,$$

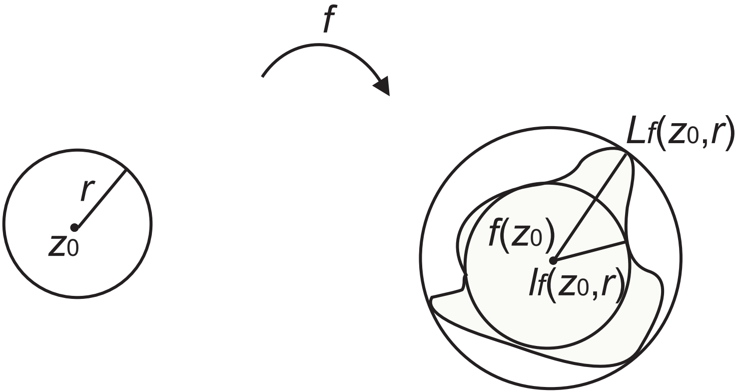
where $C = \pi e^2 \varphi_\infty(z_0)$ *and*

(9) $$\varphi_\infty(z_0) = \sup_{R\in(e,+\infty)} \frac{1}{\pi R^2} \int_{B(z_0,R)} \varphi(z)\,dxdy.$$

Asymptotic behavior at infinity

Let $z_0 \in \mathbb{C}$ and $r > 0$. For homeomorphism $f : \mathbb{C} \to \mathbb{C}$, we put

$$L_f(z_0, r) = \max_{|z-z_0|=r} |f(z) - f(z_0)|, \quad l_f(z_0, r) = \min_{|z-z_0|=r} |f(z) - f(z_0)|.$$



Asymptotic behavior at infinity

Theorem 1. *Let* μ *and* $\nu\colon \mathbb{C}\to\mathbb{C}$ *be a measurable functions with* $|\mu(z)|+|\nu(z)|<1$ *a.e. and* $f\colon \mathbb{C}\to\mathbb{C}$ *be a homeomorphic* $W^{1,1}_{loc}$ *solution of the Beltrami equation* (1), $z_0\in\mathbb{C}$. *Assume that* $K_{\mu,\nu}\in GFMV(\mathbb{C})$ *and*

$$(10)\qquad k_\infty = k_\infty(z_0) = \sup_{R\in(e,+\infty)} \frac{1}{\pi R^2}\int\limits_{B(z_0,R)} K_{\mu,\nu}(z)\,dxdy,$$

then

$$(11)\qquad \liminf_{R\to\infty}\frac{L_f(z_0,R)}{R^p}\geqslant c\, l_f(z_0,e),$$

where $p=\frac{2}{e^2k_\infty}$ *and* $c=e^{-\frac{4}{e^2k_\infty}}$.

Picking $\nu(z) \equiv 0$ in Theorem 1, we arrive at the following statement.

Corollary 2. *Let* $\mu : \mathbb{C} \to \mathbb{C}$ *be a measurable function with* $|\mu(z)| < 1$ *a.e. and* $f : \mathbb{C} \to \mathbb{C}$ *be a homeomorphic* $W^{1,1}_{loc}$ *solution of the Beltrami equation* (3), $z_0 \in \mathbb{C}$. *Assume that* $K_\mu \in GFMV(\mathbb{C})$ *and*

(12) $$k_\infty = \sup_{R \in (e,+\infty)} \frac{1}{\pi R^2} \int_{B(z_0,R)} K_\mu(z)\, dxdy,$$

then

(13) $$\liminf_{R \to \infty} \frac{L_f(z_0,R)}{R^p} \geqslant c\, l_f(z_0,e),$$

where $p = \frac{2}{e^2 k_\infty}$ *and* $c = e^{-\frac{4}{e^2 k_\infty}}$.

Example. Consider the equation

$$f_{\overline{z}} = \mu(z) f_z, \tag{14}$$

where

$$\mu(z) = \begin{cases} \frac{1-\log|z|}{1+\log|z|}\frac{z}{\overline{z}}, & |z| > e, \\ 0, & |z| \leqslant e, \end{cases} \tag{15}$$

in the complex plane $\mathbb{C}$. Hence,

$$K_\mu(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|} = \begin{cases} \log|z|, & |z| > e, \\ 1, & |z| \leqslant e \end{cases} \tag{16}$$

and

$$\limsup_{R\to\infty} \frac{1}{\pi R^2} \int\limits_{B(z_0,R)} K_\mu(z)\, dxdy = \infty, \quad z_0 = 0. \tag{17}$$

It follows that the function $K_\mu(z)$ does not have a global finite mean value at the point $z_0 = 0$.

Asymptotic behavior at infinity

The mapping

(18)
$$f = \begin{cases} \frac{\log|z|}{|z|} z, & |z| > e, \\ e^{-1} z, & |z| \leqslant e \end{cases}$$

is a solution of the equation (14).
On the other hand, we have $L_f(z_0, R) = \max_{|z|=R} |f(z)| = \log R$ as $R \geqslant e$ and

(19)
$$\lim_{R \to \infty} \frac{\max_{|z|=R} |f(z)|}{R^\beta} = 0$$

for every $\beta > 0$.

Asymptotic behavior at infinity

Letting $\mu(z) \equiv 0$ in Theorem 1, we derive the following statement.

Theorem 2. *Let $\nu \colon \mathbb{C} \to \mathbb{C}$ be a measurable function with $|\nu(z)| < 1$ a.e. and $f \colon \mathbb{C} \to \mathbb{C}$ be a homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution of the Beltrami equation* (5)*, $z_0 \in \mathbb{C}$. Assume that $K_\nu \in GFMV(\mathbb{C})$ and*

$$k_\infty = \sup_{R \in (e, +\infty)} \frac{1}{\pi R^2} \int\limits_{B(z_0, R)} K_\nu(z)\, dxdy, \tag{20}$$

then

$$\liminf_{R \to \infty} \frac{L_f(z_0, R)}{R^p} \geqslant c\, l_f(z_0, e), \tag{21}$$

where $p = \frac{2}{e^2 k_\infty}$ and $c = e^{-\frac{4}{e^2 k_\infty}}$.

Reduced Beltrami equation

Let $\lambda : D \to \mathbb{C}$ be a measurable function with $|\lambda(z)| < 1$ a.e. in D. The equation of the form

$$f_{\bar{z}} = \lambda(z) \mathrm{Re} f_z \tag{22}$$

is called the *reduced Beltrami equation*, see [8], [9], [10], [11], [12].

[8] K. Astala, T. Iwaniec, and G. J. Martin, *Elliptic Differential Equations and Quasiconformal Mappings in the Plane.* Princeton Mathematical Series, **48**, Princeton University Press, Princeton, 2009.

[9] B. Bojarski, "Generalized solutions of a system of differential equations of the first order of the elliptic type with discontinuous coefficients," *Mat. Sb.,* ***43(85)****(4), 451–503 (1957) [in Russian].*

[10] B. Bojarski, *Generalized Solutions of a System of Differential Equations of the First Order of Elliptic Type with Discontinuous Coefficients.* University Printing House, Jyväskylä, 2009.

[11] B. Bojarski, "Primary solutions of general Beltrami equations," *Ann. Acad. Sci. Fenn. Math.,* ***32****(2), 549–557 (2007).*

[12] L. I. Volkovyskii, *Quasiconformal Mappings.* L'vov University Press, L'vov, 1954 [in Russian].

Reduced Beltrami equation

Equation (22) can be rewritten as the equation (1) with

(23) $$\mu(z) = \nu(z) = \frac{\lambda(z)}{2}$$

and then

(24) $$K_{\mu,\nu}(z) = K_{\lambda}(z) = \frac{1 + |\lambda(z)|}{1 - |\lambda(z)|}.$$

Reduced Beltrami equation

Thus, the previous results give the following consequence for the reduced Beltrami equations (22).

Corollary 3. *Let* $\lambda : \mathbb{C} \to \mathbb{C}$ *be a measurable function with* $|\lambda(z)| < 1$ *a.e. and* $f : \mathbb{C} \to \mathbb{C}$ *be a homeomorphic* $W^{1,1}_{loc}$ *solution of the reduced Beltrami equation* (22), $z_0 \in \mathbb{C}$. *Assume that* $K_\lambda \in GFMV(\mathbb{C})$ *and*

$$k_\infty = \sup_{R \in (e, +\infty)} \frac{1}{\pi R^2} \int\limits_{B(z_0, R)} K_\lambda(z)\, dxdy, \tag{25}$$

then

$$\liminf_{R \to \infty} \frac{L_f(z_0, R)}{R^p} \geqslant c\, l_f(z_0, e), \tag{26}$$

where $p = \frac{2}{e^2 k_\infty}$ *and* $c = e^{-\frac{4}{e^2 k_\infty}}$.

Non-existence theorems

We find sufficient conditions under which the Beltrami equation with two characteristics has no homeomorphic solutions in the Sobolev class $W^{1,1}_{loc}$ with the given asymptotic conditions.

Theorem 3. *Let* μ *and* $\nu : \mathbb{C} \to \mathbb{C}$ *be a measurable functions with* $|\mu(z)| + |\nu(z)| < 1$ *a.e.,* $z_0 \in \mathbb{C}$*. If* $K_{\mu,\nu} \in GFMV(\mathbb{C})$*, then there are no homeomorphic solutions* $f : \mathbb{C} \to \mathbb{C}$ *of the equation* (1) *from Sobolev class* $W^{1,1}_{loc}$ *satisfying the asymptotic condition*

(27) $$\liminf_{R\to\infty} \frac{L_f(z_0, R)}{R^p} = 0,$$

where $p = \frac{2}{e^2 k_\infty}$ *and* $k_\infty = \sup\limits_{R\in(e,+\infty)} \frac{1}{\pi R^2} \int\limits_{B(z_0,R)} K_{\mu,\nu}(z)\, dxdy$*.*

Non-existence theorems

Picking in Theorem 3 $\nu(z) \equiv 0$, we get

Corollary 4. *Let* $\mu : \mathbb{C} \to \mathbb{C}$ *be a measurable function with* $|\mu(z)| < 1$ *a.e.,* $z_0 \in \mathbb{C}$. *If* $K_\mu \in GFMV(\mathbb{C})$, *then there are no homeomorphic solutions* $f : \mathbb{C} \to \mathbb{C}$ *of the equation* (3) *from Sobolev class* $W^{1,1}_{loc}$ *satisfying the asymptotic condition*

(28) $$\liminf_{R \to \infty} \frac{L_f(z_0, R)}{R^p} = 0,$$

where $p = \frac{2}{e^2 k_\infty}$ *and* $k_\infty = \sup_{R \in (e, +\infty)} \frac{1}{\pi R^2} \int_{B(z_0,R)} K_\mu(z)\, dxdy.$

Non-existence theorems

Letting $\mu(z) \equiv 0$ in Theorem 3 gives

Corollary 5. *Let* $\nu : \mathbb{C} \to \mathbb{C}$ *be a measurable function with* $|\nu(z)| < 1$ *a.e.,* $z_0 \in \mathbb{C}$*. If* $K_\nu \in GFMV(\mathbb{C})$*, then there are no homeomorphic solutions* $f : \mathbb{C} \to \mathbb{C}$ *of the equation* (5) *from Sobolev class* $W^{1,1}_{loc}$ *satisfying the asymptotic condition*

(29) $$\liminf_{R\to\infty} \frac{L_f(z_0, R)}{R^p} = 0,$$

where $p = \frac{2}{e^2 k_\infty}$ *and* $k_\infty = \sup\limits_{R\in(e,+\infty)} \frac{1}{\pi R^2} \int\limits_{B(z_0,R)} K_\nu(z)\, dxdy.$

Non-existence theorems

Now choosing $\mu(z) = \nu(z) = \frac{\lambda(z)}{2}$ in Theorem 3 provides

Corollary 6. *Let* $\lambda : \mathbb{C} \to \mathbb{C}$ *be a measurable function with* $|\lambda(z)| < 1$ *a.e.,* $z_0 \in \mathbb{C}$. *If* $K_\lambda \in GFMV(\mathbb{C})$, *then there are no homeomorphic solutions* $f : \mathbb{C} \to \mathbb{C}$ *of the equation* (22) *from Sobolev class* $W^{1,1}_{loc}$ *satisfying the asymptotic condition*

(30)
$$\liminf_{R\to\infty} \frac{L_f(z_0, R)}{R^p} = 0,$$

where $p = \frac{2}{e^2 k_\infty}$ *and* $k_\infty = \sup\limits_{R\in(e,+\infty)} \frac{1}{\pi R^2} \int\limits_{B(z_0,R)} K_\lambda(z)\, dxdy.$