

# Horizontal Gauss maps of hypersurfaces in 2-step Carnot groups

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Let  $G$  be a  $(n + 1)$ -dimensional Lie group with a left invariant metric  $\langle \cdot, \cdot \rangle$ . For an (immersed) oriented hypersurface  $M$  in  $G$  with the unit normal field  $\eta$  define the (*left-invariant*) *Gauss map*  $\Phi: M \rightarrow S^n$  of  $M$  by

$$\Phi(p) = dL_{p^{-1}}(\eta(p)),$$

where a point  $p \in M$  is identified with its image under the immersion,  $dL_{p^{-1}}$  is the differential of the left translation in  $G$ , and  $S^n$  is the unit sphere in the Lie algebra  $\mathfrak{g}$  of  $G$  with respect to  $\langle \cdot, \cdot \rangle$ . This is a generalization of the Gauss map of a hypersurface in the Euclidean space, that is harmonic if and only if  $M$  is of constant mean curvature (CMC) by a well-known result of E. A. Ruh and J. Vilms [5]. This equivalence does not take place in general. For example, each CMC hypersurface with the harmonic Gauss map in the  $(2m + 1)$ -dimensional Heisenberg group is locally a vertical cylinder [4]. Note that for a Riemannian manifold  $M$  a map  $\Phi: M \rightarrow S^n$  is harmonic if and only if

$$\Delta\Phi = 2e(\Phi)\Phi,$$

where  $\Delta$  is the Riemannian Laplacian of  $M$ ,

$$\Delta\Phi = (\Delta\Phi^1, \dots, \Delta\Phi^{n+1})$$

for  $\Phi = (\Phi^1, \dots, \Phi^{n+1})$ , and  $e(\Phi)$  is the energy density function of  $\Phi$ .

Let now  $G$  be a *2-step nilpotent group*. This means that for the (non-trivial) center  $\mathfrak{z}$  of  $\mathfrak{g}$  and the orthogonal complement  $\mathfrak{h}$  of  $\mathfrak{z}$  with respect to  $\langle \cdot, \cdot \rangle$  we have  $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{z}$ . Assume additionally that  $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{z}$ , i.e.,  $G$  is a *2-step Carnot group*. Then the left-invariant distribution  $\mathcal{H}$  generated by  $\mathfrak{h}$  is totally non-integrable and, together with the restriction of  $\langle \cdot, \cdot \rangle$  to  $\mathcal{H}$ , defines a sub-Riemannian structure on  $G$ . For any  $p \in G$  let us call elements of  $\mathcal{H}_p$  *horizontal*, and elements of  $\mathcal{Z}_p$ , where  $\mathcal{Z}$  is the left-invariant distribution generated by  $\mathfrak{z}$ , *vertical*.

Let a neighborhood  $U \subset M$  consist of non-singular points, i.e.  $\mathcal{H}_p \not\subset T_pM$  for any  $p \in U$ . Then the intersections  $\mathcal{H}_p \cap T_pM$  form a  $q$ -dimensional distribution on  $U$ , where

$$\dim \mathfrak{h} = q + 1 < n + 1.$$

Together with the restriction of the first fundamental form of  $M$ , it yields a sub-Riemannian structure on  $U$ . Let  $\Delta_H$  be a sub-Riemannian Laplacian of this structure [1]. For each  $p \in U$  the orthogonal projection  $\eta(p)^{\mathcal{H}}$  of  $\eta(p)$  to  $\mathcal{H}_p$  is non-zero, so the *horizontal unit normal field*

$$\eta_H: p \mapsto \eta(p)^{\mathcal{H}} / |\eta(p)^{\mathcal{H}}|$$

is well defined on  $U$ . Similarly to the previous definition of the Gauss map, we can then call

$$\Phi_H: U \rightarrow S^q: p \mapsto dL_{p^{-1}}(\eta_H(p))$$

the (*left-invariant*) *horizontal Gauss map* to the unit sphere of  $\mathfrak{h}$ .

The *horizontal second fundamental form*  $b_H$  on  $U$  is constructed as follows [3]. For any two horizontal tangent vector fields  $X, Y$  on  $U$ , i.e.,  $X(p), Y(p) \in \mathcal{H}_p \cap T_pM$  for any  $p \in U$ ,

$$b_H(X, Y) = \langle \nabla_X Y, \eta_H \rangle,$$

where  $\nabla$  is the Riemannian connection of  $\langle \cdot, \cdot \rangle$ . Let  $\{E_k\}_{k=1}^q$  be a local orthonormal frame of the distribution  $p \mapsto \mathcal{H}_p \cap T_pM$ . Then

$$\sum_{k=1}^q b_H(E_k, E_k),$$

i.e., the trace of  $b_H$  with respect to  $\langle \cdot, \cdot \rangle$ , is called the *horizontal mean curvature* of  $U$ , and

$$|b_H|^2 = \sum_{k,l=1}^q b_H(E_k, E_l)^2$$

is the *square norm* of  $b_H$  of  $U$  with respect to  $\langle \cdot, \cdot \rangle$ .

**Theorem 1.** *Let  $p$  be a non-singular point of  $M$  and  $U \subset M$  be a neighborhood of  $p$  consisting of non-singular points. Let  $\eta(p) = \lambda X_{q+1} + \mu Z_{n-q}$ , where  $X_{q+1}$  and  $Z_{n-q}$  are unit horizontal and vertical vectors, respectively, and let  $\{X_k\}_{k=1}^q$  be an orthonormal basis of  $\mathcal{H}_p \cap T_p M$ . Denote also by  $X_k$  left-invariant vector fields extending these vectors and corresponding to elements of  $\mathfrak{h}$ . Then*

$$\Delta_H \Phi_H(p) = \sum_{k=1}^q \left( -X_k(H) + \text{Ric}(X_k, X_{q+1}) \right) X_k - \left( |b_H|^2(p) + \frac{1}{2} \text{Ric}(X_{q+1}, X_{q+1}) \right) X_{q+1},$$

where  $H$  is the horizontal mean curvature of  $U$  and  $\text{Ric}$  is the Ricci tensor of  $\langle \cdot, \cdot \rangle$ .

Note that  $\Phi_H(p) = X_{q+1}$ . Similarly to the harmonicity criterion for maps into spheres stated above, we will call  $\Phi_H$  *horizontally harmonic* at  $p$ , if  $\Delta_H \Phi_H(p)$  is proportional to  $X_{q+1}$ . For each  $Z \in \mathfrak{z}$  denote by

$$J(Z): \mathfrak{h} \rightarrow \mathfrak{h}$$

the linear operator defined by

$$\langle J(Z)X, Y \rangle = \langle [X, Y], Z \rangle$$

for any  $X, Y \in \mathfrak{h}$ . It is said that  $G$  is of *Heisenberg type* [2], if

$$J(Z)^2 = -|Z|^2 Id_{\mathfrak{h}}.$$

The  $(2m+1)$ -dimensional Heisenberg group is an example of such a group. For a group  $G$  of Heisenberg type and any  $X, Y \in \mathfrak{h}$  we have

$$\text{Ric}(X, Y) = -\frac{1}{2}(n-q)\langle X, Y \rangle,$$

therefore Theorem 1 implies the following analogue of the Ruh–Vilms theorem:

**Corollary 2.** *Let  $G$  be of Heisenberg type. Then the horizontal mean curvature of  $U$  is constant in all horizontal directions if and only if the horizontal Gauss map of  $U$  is horizontally harmonic.*

We will also discuss possible applications of these results.

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