On R-curvature tensor of locally conformal C_{12}-manifolds

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Definition 1 ([3]). An almost contact metric manifold $(M^{2n+1}, g, \xi, \eta, \Phi)$ is said to be a locally conformal C_{12}-manifold if the following identity holds:

$$\begin{aligned}\nabla_X(\Phi)Y = & -g(X,\Phi Y)\,\alpha^{\#} + g(X,Y)\,\Phi(\alpha^{\#}) + \mathrm{d}\alpha(\Phi Y)\,X - \mathrm{d}\alpha(Y)\,\Phi X \\ & - \eta(X)\Big\{\eta(Y)\Big[\Phi(\nabla_\xi \xi) + \Phi(\alpha^{\#})\Big] + g(\nabla_\xi\xi, \Phi Y)\,\xi + g(\alpha^{\#}, \Phi Y)\,\xi\Big\},\end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$, where $\alpha^{\#} = \operatorname{grad}\alpha$ is the vector field described by

$$g(\alpha^{\#}, X) = X(\alpha) = \mathrm{d}\alpha(X).$$

Theorem 2 ([1]). *Let $\{\omega^i\}$ be a local orthonormal coframe defined on the open set $U \subset M$; then on U there exist unique 1-forms $\{\theta^i_j\}$, $i, j = 0, 1, \dots, 2n$, such that the second structure equation is given by*

$$\mathrm{d}\theta^i_j = -\theta^i_k \wedge \theta^k_j + \tfrac{1}{2}\,R^i_{jkl}\,\omega^k \wedge \omega^l.$$

Proposition 3. *If M^{2n+1} is a locally conformal C_{12}-manifold, then on M we have*

(1) $\theta^{\hat a}_b = (\alpha^b\,\delta^a_c - \alpha^a\,\delta^b_c)\,\omega^c$;
(2) $\theta^b_{\hat a} = (\alpha_b\,\delta^c_a - \alpha_a\,\delta^c_b)\,\omega_c$;
(3) $\theta^a_0 = C^a\,\omega + \alpha_0\,\omega^a$;
(4) $\theta^{\hat a}_0 = C_a\,\omega + \alpha_0\,\omega_a$;
(5) $\theta^0_a = -C_a\,\omega - \alpha_0\,\omega_a$;
(6) $\theta^0_{\hat a} = -C^a\,\omega - \alpha_0\,\omega^a$,

where $a, b, c = 1, 2, \dots, n$, and $\hat a = a + n$.

Theorem 4. *The Riemannian curvature tensor of a locally conformal C_{12}-manifold M^{2n+1}, with $n > 1$, has the following components on the G-structure adjoined space:*

(1) $R^a_{0\hat h 0} = C^{ah} - C^a C^h$,
(2) $R^a_{0h0} = C^a{}_h - C^a C_h - \alpha_{00}\,\delta^a_h - (\alpha_0)^2\,\delta^a_h - 2\alpha^{[a}\,\delta^{d]}_h\,C_d$,
(3) $R^a_{0hd} = 2\alpha_{0[h}\,\delta^a_{d]}$,
(4) $R^a_{0h\hat d} = -\alpha_0{}^d\,\delta^a_h + 2\alpha_0\,\delta^a_b\,\alpha^{[b}\,\delta^{d]}_h - 2\alpha^{[a}\,\delta^{c]}_h\,\alpha_0\,\delta^d_c$,
(5) $R^a_{bhd} = 2A^a_{bhd}$,
(6) $R^a_{bh\hat d} = A^{ad}_{bh} - (\alpha_0)^2\,\delta^a_h\,\delta^d_b + 4\alpha^{[a}\,\delta^{c]}_h\,\alpha_{[c}\,\delta^d_{b]}$,
(7) $R^a_{bh0} = A^a_{bh0} - \alpha_0\,C_b\,\delta^a_h$,
(8) $R^a_{b\hat h\hat d} = 2A^{ahd}_b$,
(9) $R^a_{b\hat h 0} = A^{ah0}_b + \alpha_0\,C^a\,\delta^h_b$,
(10) $R^a_{\hat b hd} = 2\left(\alpha^b{}_{[h}\,\delta^a_{d]} - \alpha^a{}_{[h}\,\delta^b_{d]} - (\alpha_0)^2\,\delta^a_{[h}\,\delta^b_{d]}\right)$,
(11) $R^a_{\hat b h\hat d} = -\left(\alpha^{bd}\,\delta^a_h - \alpha^{ad}\,\delta^b_h + 4\alpha^{[a}\,\delta^{b]}_c\,\alpha^{[c}\,\delta^{d]}_h\right)$,
(12) $R^a_{\hat b h0} = -\alpha^{b0}\,\delta^a_h + \alpha^{a0}\,\delta^b_h + 2\alpha^{[a}\,\delta^{b]}_h\,\alpha_0 + \alpha_0\,C^a\,\delta^b_h - \alpha_0\,C^b\,\delta^a_h$,

and all other components are zero or can be found by the symmetry properties or conjugates of the above components.

Theorem 5. *On the G-structure adjoined space, the Ricci tensor components for a locally conformal C_{12}-manifold are given by*

(1) $r_{00} = C^a{}_a + C_a{}^a - 2C^a\,C_a - 2n\left(\alpha_{00} + (\alpha_0)^2\right) + (n-1)\,\alpha^d\,C_d + (n-1)\,\alpha_d\,C^d$,

(2) $r_{a0} = A^b_{ab0} - (2n-1)\,\alpha_0\, C_a - (n-1)\,\alpha_{a0} - (n-1)\,\alpha_0\,\alpha_a,$
(3) $r_{ab} = C_{ab} - C_a\, C_b - 2A^c_{abc} - 2(n-1)\,\alpha_{ab},$
(4) $r_{\hat{a}b} = C^a{}_b - C^a\, C_b + \big(-\alpha_{00} - 2n(\alpha_0)^2 + \alpha^d\, C_d - \alpha^c{}_c - (n-1)\,\alpha^h\,\alpha_h\big)\,\delta^a_b$
$+ (n-1)\,\alpha^a\,\alpha_b - (n-2)\,\alpha^a{}_b + A^{ac}_{cb} - \alpha^a\, C_b,$

and the remaining components are determined by symmetry, or can be found by using the conjugates of the given components.

Theorem 6. *A locally conformal C_{12}-manifold is an η-Einstein manifold if and only if the following conditions hold:*
(1) $\lambda = C^a{}_a + C_a{}^a - 2C^a\, C_a - 2n\left(\alpha_{00} + (\alpha_0)^2\right) + (n-1)\,\alpha^d\, C_d + (n-1)\,\alpha_d\, C^d - \mu,$
(2) $A^b_{ab0} = (2n-1)\,\alpha_0\, C_a + (n-1)\,\alpha_{a0} + (n-1)\,\alpha_0\,\alpha_a,$
(3) $A^c_{abc} = \frac{1}{2}\,(C_{ab} - C_a\, C_b) - (n-1)\,\alpha_{ab},$
(4) $A^{ac}_{cb} = \big(C^a{}_a + C_a{}^a - 2C^a\, C_a - (2n-1)\,\alpha_{00} + (n-2)\,\alpha^d\, C_d + (n-1)\,\alpha_d\, C^d + \alpha^c{}_c$
$+ (n-1)\,\alpha^h\,\alpha_h - \mu\big)\,\delta^a_b - C^a{}_b + C^a\, C_b + \alpha^a\, C_b + (n-2)\,\alpha^a{}_b - (n-1)\,\alpha^a\,\alpha_b.$

Definition 7 ([2])**.** An almost contact metric manifold $(M^{2n+1}, g, \xi, \eta, \Phi)$ is called a manifold of class CR_1 if the curvature tensor R satisfies

$$g(R(\Phi^2X, \Phi^2Y)\,\Phi Z, \Phi W) = g(R(\Phi X, \Phi Y)\,\Phi Z, \Phi W), \quad \forall\, X, Y, Z, W \in \mathfrak{X}(M).$$

Whereas M is called a manifold of class CR_2 if the curvature tensor R satisfies

$$g(R(\Phi^2X, \Phi^2Y)\,\Phi^2Z, \Phi^2W) = g(R(\Phi X, \Phi Y)\,\Phi^2Z, \Phi^2W) + g(R(\Phi X, \Phi^2Y)\,\Phi Z, \Phi^2W) + g(R(\Phi X, \Phi^2Y)\,\Phi^2Z, \Phi W), \quad \forall\, X, Y, Z, W \in \mathfrak{X}(M).$$

On the other hand, M is called a manifold of class CR_3 if the curvature tensor R satisfies

$$g(R(\Phi^2X, \Phi^2Y)\,\Phi^2Z, \Phi^2W) = g(R(\Phi X, \Phi Y)\,\Phi Z, \Phi W), \quad \forall\, X, Y, Z, W \in \mathfrak{X}(M).$$

Proposition 8. *A locally conformal C_{12}-manifold M is a CR_3-manifold if and only if $A^a_{bcd} = 0$ on the G-structure adjoined space.*

Proposition 9. *A locally conformal C_{12}-manifold M is a CR_2-manifold if and only if $A^a_{bcd} = 0$ on the G-structure adjoined space.*

Corollary 10. *A locally conformal C_{12}-manifold M is a CR_2-manifold if and only if M is a CR_3-manifold. That is, on M, $CR_2 = CR_3$.*

Proposition 11. *A locally conformal C_{12}-manifold M is a CR_1-manifold if and only if $A^a_{bcd} = 0$ and $\alpha^a{}_c = -\frac{1}{2}(\alpha_0)^2\,\delta^a_c$ on the G-structure adjoined space.*

Theorem 12. *Let M be a locally conformal C_{12}-manifold of class CR_3. Then M is an η-Einstein manifold if and only if the following conditions are satisfied:*
(1) $\lambda = C^a{}_a + C_a{}^a - 2C^a\, C_a - 2n\left(\alpha_{00} + (\alpha_0)^2\right) + (n-1)\,\alpha^d\, C_d + (n-1)\,\alpha_d\, C^d - \mu,$
(2) $A^b_{ab0} = (2n-1)\,\alpha_0\, C_a + (n-1)\,\alpha_{a0} + (n-1)\,\alpha_0\,\alpha_a,$
(3) $\alpha_{ab} = \dfrac{1}{2(n-1)}\,(C_{ab} - C_a\, C_b)$ *$(n > 1)$ and* $\alpha_{[ab]} = 0,$
(4) $A^{ac}_{cb} = \big(C^a{}_a + C_a{}^a - 2C^a\, C_a - (2n-1)\,\alpha_{00} + (n-2)\,\alpha^d\, C_d + (n-1)\,\alpha_d\, C^d + \alpha^c{}_c$
$+ (n-1)\,\alpha^h\,\alpha_h - \mu\big)\,\delta^a_b - C^a{}_b + C^a\, C_b + \alpha^a\, C_b + (n-2)\,\alpha^a{}_b - (n-1)\,\alpha^a\,\alpha_b.$

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