Sub-Hamiltonian Framework for Sub-Finsler Geometry

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A Hamiltonian space is a pair $(M, \tilde{H})$, where M is an n-dimensional manifold and

$$\tilde{H} : T^*M \to \mathbb{R}$$

is a smooth Hamiltonian function with nondegenerate Hessian determinant. It was introduced and studied by R. Miron [5]. Hamiltonian geometry is the branch of differential geometry that studies Hamiltonian spaces, Hamiltonian systems, symplectic structures, canonical flows, connections, curvature, Hamiltonian vector fields, and related geometric-dynamical properties. It plays a fundamental role in various areas of mathematics and physics. The works of R. Miron and his co-authors have made significant contributions to the development and advancement of Hamiltonian geometry, providing valuable insights into the geometric foundations of Hamiltonian systems [6]. Hamiltonian geometry provides a natural framework for studying geodesics, optimal control, and dynamical structures in sub-Riemannian and sub-Finsler geometry, particularly through Hamiltonian flows and Legendre duality; see [1, 7] for more details.

We develop sub-Finsler bundle geometry and its relationship with sub-Hamiltonian functions on a sub-Finsler manifold $(M, \mathcal{D}, F)$. The sub-Hamiltonian $H : \mathcal{D}^* \to \mathbb{R}$ is positive semi-definite, vanishing only at $p = 0$. Also, we establish a three-way equivalence characterizing forward geodesic completeness via global sub-Hamiltonian flows and compactness of forward distance balls, as a consequence of the sub-Finsler Hopf–Rinow theorem. Moreover, under completeness and controllability, existence of a time-optimal control is proved through properness of the sub-Finsler distance. The Legendre transformation reduces energy minimization to a first-order ODE on $\mathcal{D}^*$, yielding normal geodesics as solutions. Furthermore, three dynamical results follow: geodesics are integral curves of $\vec{H}$; the Liouville volume form on T^*M is preserved under the flow of $\vec{H}$ via canonical extension of H; and $\vec{H}$ is symplectically orthogonal to all $\mathcal{D}$-preserving vector fields in $\mathcal{D}_x^0$, encoding nonholonomic constraints geometrically. These results establish a rigorous Hamiltonian framework for sub-Finsler geometry with direct applications to nonholonomic mechanics, robotics, and optimal control. Lastly, abnormal extremals and three open problems are identified.

References

[1] Andrei Agrachev, Davide Barilari, and Ugo Boscain. *A Comprehensive Introduction to Sub-Riemannian Geometry*. Number 181 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2019. doi:10.1017/9781108677325.

[2] Andrei A. Agrachev and Yuri L. Sachkov. *Control Theory from the Geometric Viewpoint*, volume 87 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. doi:10.1007/978-3-662-06404-7.

[3] Layth M. Alabdulsada. Sub-Finsler geometry and nonholonomic mechanics. *Rocky Mountain J. Math.*, 55(3):597–617, 2025. doi:10.1216/rmj.2025.55.597.

[4] Layth M. Alabdulsada and László Kozma. On the connection of sub-Finslerian geometry. *Int. J. Geom. Methods Mod. Phys.*, 16(supp02):1941006, 2019. doi:10.1142/S0219887819410068.

[5] Radu Miron. Sur la géométrie des espaces Hamilton. *C. R. Acad. Sci. Paris Sér. II*, 306(4):195–198, 1988.

[6] Radu Miron. Hamilton geometry. *An. Şt. Univ. "Al. I. Cuza" Iaşi*, 35:33–85, 1989.

[7] Richard Montgomery. *A Tour of Subriemannian Geometries, Their Geodesics and Applications*, volume 91 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002. doi:10.1090/surv/091.