

# Sturm–Liouville problems with a boundary condition depending linearly on an eigenparameter

**Yagub N. Aliyev**

ADA University, School of IT and Engineering, 61 Ahmadbay Agha-Oglu Street, Baku AZ1008,  
Azerbaijan

*E-mail:* yaliyev@ada.edu.az

**Narmin N. Aliyeva**

Baku State University, Department of Differential Equations and Control Theory, Academic Zahid  
Khalilov Str. 23, Baku AZ1148, Azerbaijan

*E-mail:* nermin.aliyeva@idrak.edu.az

Consider the following spectral problem

$$-y'' + q(x)y = \lambda y, \quad 0 < x < 1, \quad (1)$$

$$y(0) \cos \beta = y'(0) \sin \beta, \quad 0 \leq \beta < \pi, \quad (2)$$

$$(a\lambda + b)y(1) = (c\lambda + d)y'(1), \quad (3)$$

where  $\lambda$  is a spectral parameter,  $q(x)$  is a real-valued and continuous function over the interval  $(0, 1)$ , and  $a, b, c, d$  are real constants with  $ad - bc \neq 0$ . It is possible to find the eigenfunctions and associated functions corresponding to the eigenvalues  $\lambda_n$ .

The sufficiently large eigenvalues of the boundary value problem (1)–(3) are real, simple, and form an infinite increasing sequence

$$\lambda_0 < \lambda_1 < \lambda_2 < \dots$$

Let  $\varphi(x, \lambda)$  be a non-zero solution of (1)–(2). We write the characteristic function, which is analytic, as

$$\omega(\lambda) = (a\lambda + b)y(1, \lambda) - (c\lambda + d)y'(1, \lambda).$$

It follows from the boundary condition (3) that a necessary and sufficient condition for  $\lambda$  to be an eigenvalue is that the equation

$$\omega(\lambda) = 0$$

is satisfied. If, in addition, the inequality

$$\omega'(\lambda) \neq 0$$

is satisfied, then we say that  $\lambda$  is a simple eigenvalue.

It is known that the boundary value problem (1)–(3) has infinitely many eigenvalues, and the following cases are possible:

- (a) all eigenvalues are real and simple;
- (b) all eigenvalues are real and all, except one double  $\lambda_k$ , are simple;
- (c) all eigenvalues are real and all, except one triple  $\lambda_k$ , are simple;
- (d) all eigenvalues are simple and all, except a conjugate pair of non-real ones  $\lambda_s = \overline{\lambda_r}$ , are real.

The eigenfunctions and associated functions are defined as follows. If  $\lambda_k$  is a double eigenvalue ( $\lambda_k = \lambda_{k+1}$ ) or if  $\lambda_k$  is a triple eigenvalue ( $\lambda_k = \lambda_{k+1} = \lambda_{k+2}$ ), then for the associated function  $y_{k+1}$  corresponding to the eigenfunction  $y_k$ , the following relations hold:

$$-y''_{k+1} + q(x)y_{k+1} = \lambda_k y_{k+1} + y_k, \quad (4)$$

$$y_{k+1}(0) \cos \beta = y'_{k+1}(0) \sin \beta, \quad (5)$$

$$(a\lambda_k + b)y_{k+1}(1) + a y_k(1) = (c\lambda_k + d)y'_{k+1}(1) + c y'_k(1). \quad (6)$$

If  $\lambda_k$  is an eigenvalue of multiplicity three, i.e.,  $\lambda_k = \lambda_{k+1} = \lambda_{k+2}$ , then in addition to the first-order associated function  $y_{k+1}$  defined by (4)–(6), there exists a second-order associated function  $y_{k+2}$ :

$$-y''_{k+2} + q(x)y_{k+2} = \lambda_k y_{k+2} + y_{k+1}, \quad (7)$$

$$y_{k+2}(0) \cos \beta = y'_{k+2}(0) \sin \beta, \quad (8)$$

$$(a\lambda_k + b)y_{k+2}(1) + a y_{k+1}(1) = (c\lambda_k + d)y'_{k+2}(1) + c y'_{k+1}(1). \quad (9)$$

Let  $y_{k+1}^*$ ,  $y_{k+1}^\#$ , and  $y_{k+2}^\#$  be functions satisfying the following relations:

$$(y_{k+1}^*, y_{k+1}) = -(ad - bc) \mathfrak{A}(y_{k+1}^*) \mathfrak{A}(y_{k+1}),$$

$$\begin{aligned} \left( y_{k+1}^\#, y_{k+2} \right) &= -(ad - bc) \mathfrak{A}(y_{k+1}^\#) \mathfrak{A}(y_{k+2}), \\ \left( y_{k+2}^\#, y_{k+2} \right) &= -(ad - bc) \mathfrak{A}(y_{k+2}^\#) \mathfrak{A}(y_{k+2}). \end{aligned}$$

Let us define:

$$\begin{aligned} \mathfrak{A}(y_{k+1}^*) &= \begin{cases} \frac{y_{k+1}^*(1)}{c\lambda_k + d} - \frac{c y_k(1)}{(c\lambda_k + d)^2}, & \lambda_k \neq -\frac{d}{c}, \\ \frac{(y_{k+1}^*)'(1)}{a\lambda_k + b} - \frac{a y_k'(1)}{(a\lambda_k + b)^2}, & \lambda_k = -\frac{d}{c}, \end{cases} \\ \mathfrak{A}(y_{k+1}^\#) &= \begin{cases} \frac{y_{k+1}^\#(1)}{c\lambda_k + d} - \frac{c y_k(1)}{(c\lambda_k + d)^2}, & \lambda_k \neq -\frac{d}{c}, \\ \frac{(y_{k+1}^\#)'(1)}{a\lambda_k + b} - \frac{a y_k'(1)}{(a\lambda_k + b)^2}, & \lambda_k = -\frac{d}{c}, \end{cases} \\ \mathfrak{A}(y_{k+2}^\#) &= \begin{cases} \frac{y_{k+2}^\#(1)}{c\lambda_k + d} - \frac{c y_{k+1}^\#(1)}{(c\lambda_k + d)^2} + \frac{c^2 y_k(1)}{(c\lambda_k + d)^3}, & \lambda_k \neq -\frac{d}{c}, \\ \frac{(y_{k+2}^\#)'(1)}{a\lambda_k + b} - \frac{a (y_{k+1}^\#)'(1)}{(a\lambda_k + b)^2} + \frac{a^2 y_k'(1)}{(a\lambda_k + b)^3}, & \lambda_k = -\frac{d}{c}. \end{cases} \end{aligned}$$

**Theorem 1.** Assume that all eigenvalues of problem (1)–(3) are real and simple. Then the system

$$\{y_n\} \quad (n = 0, 1, \dots; n \neq l), \quad (10)$$

where  $l$  is a non-negative integer, is a basis of the space  $L_p(0, 1)$  ( $1 < p < +\infty$ ).

**Theorem 2.** If  $\lambda_k$  is an eigenvalue of multiplicity two, then the system

$$\{y_n\} \quad (n = 0, 1, \dots; n \neq k + 1), \quad (11)$$

is a basis of the space  $L_p(0, 1)$  ( $1 < p < +\infty$ ).

**Theorem 3.** If  $\lambda_k$  is an eigenvalue of multiplicity two, then the system

$$\{y_n\} \quad (n = 0, 1, \dots; n \neq k), \quad (12)$$

is a basis of the space  $L_p(0, 1)$  ( $1 < p < +\infty$ ) if and only if  $\mathfrak{A}(y_{k+1}^*) \neq 0$ .

**Theorem 4.** If  $\lambda_k$  is an eigenvalue of multiplicity two, then the system

$$\{y_n\} \quad (n = 0, 1, \dots; n \neq l), \quad (13)$$

where  $l \neq k$ ,  $k + 1$  is a non-negative integer, is a basis of the space  $L_p(0, 1)$  ( $1 < p < +\infty$ ).

**Theorem 5.** If  $\lambda_k$  is an eigenvalue of multiplicity three, then the system

$$\{y_n\} \quad (n = 0, 1, \dots; n \neq k + 2), \quad (14)$$

is a basis of the space  $L_p(0, 1)$  ( $1 < p < +\infty$ ).

**Theorem 6.** If  $\lambda_k$  is an eigenvalue of multiplicity three, then the system

$$\{y_n\} \quad (n = 0, 1, \dots; n \neq k + 1), \quad (15)$$

is a basis of the space  $L_p(0, 1)$  ( $1 < p < +\infty$ ) if and only if  $\mathfrak{A}(y_{k+1}^\#) \neq 0$ .

**Theorem 7.** If  $\lambda_k$  is an eigenvalue of multiplicity three, then the system

$$\{y_n\} \quad (n = 0, 1, \dots; n \neq k), \quad (16)$$

is a basis of the space  $L_p(0, 1)$  ( $1 < p < +\infty$ ) if and only if  $\mathfrak{A}(y_{k+2}^\#) \neq 0$ .

**Theorem 8.** If  $\lambda_k$  is an eigenvalue of multiplicity three, then the system

$$\{y_n\} \quad (n = 0, 1, \dots; n \neq l), \quad (17)$$

where  $l \neq k$ ,  $k + 1$ ,  $k + 2$  is a non-negative integer, is a basis of the space  $L_p(0, 1)$  ( $1 < p < +\infty$ ).

**Theorem 9.** Let  $\lambda_r$  and  $\lambda_s = \overline{\lambda_r}$  be a pair of complex conjugate non-real eigenvalues. Then each of the systems

$$\{y_n\} \quad (n = 0, 1, \dots; n \neq r), \quad (18)$$

and

$$\{y_n\} \quad (n = 0, 1, \dots; n \neq l), \quad (19)$$

where  $l$  is a non-negative integer with  $l \neq r, s$ , is a basis of the space  $L_p(0, 1)$  ( $1 < p < +\infty$ ).

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