The equations of Gauss, Codazzi and Ricci of surfaces in 4-dimensional space forms

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Let N be a Riemannian, neutral or Lorentzian 4-dimensional space form with constant sectional curvature L_0. We can understand the equations of Gauss, Codazzi and Ricci of a space-like or time-like surface in N, in terms of the induced connection (of the complexification) of the two-fold exterior power of the pull-back bundle on the surface.

Suppose that N is Riemannian. Then the two-fold exterior power of the pull-back bundle on a surface in N is decomposed into two orientable subbundles of rank 3. Computing the curvature tensors of these subbundles, we obtain the expressions of the equations of Gauss, Codazzi and Ricci of the surface given in [8] ([7]). By the equations of Gauss and Ricci, the degeneracy of the twistor lifts of a surface just means that the surface satisfies $K \equiv L_0$ for the intrinsic curvature K and $R^{\perp} \equiv 0$ for the curvature tensor $R^{\perp}$ of the normal connection ([7]). We obtain a characterization of a surface such that the covariant derivative of the complex lift by $\partial/\partial w$ or $\partial/\partial \overline{w}$ for every local complex coordinate w of the surface vanishes ([7]). The twistor lifts of such a surface are degenerate.

Suppose that N is neutral. Then analogous discussions and results are valid for space-like or time-like surfaces in N. We obtain a characterization of a time-like surface in N with $L_0 \neq 0$ which has local sections of the time-like twistor spaces with fully light-like covariant derivative on a neighborhood of each point of the surface ([7]). Refer to [2], [4], [5], [6] for time-like surfaces with zero mean curvature vector such that their time-like twistor lifts have zero or light-like covariant derivative. We obtain characterizations of a time-like surface in N such that the time-like twistor lifts are horizontal along integral curves of a light-like one-dimensional distribution ([7]). In particular, a time-like surface has this property if and only if the surface has zero mean curvature vector so that the shape operator of any normal vector field is zero or light-like, and therefore its time-like twistor lifts have zero or light-like covariant derivative ([4]). We obtain a characterization of a time-like surface in N such that the covariant derivative of the paracomplex lift by $\partial/\partial \breve{w}$ or $\partial/\partial \overline{\breve{w}}$ for every local paracomplex coordinate $\breve{w}$ of the surface vanishes ([7]). If the mean curvature vector of such a surface vanishes, then a light-like normal vector field is parallel ([7]), and therefore its time-like twistor lifts have zero or light-like covariant derivative ([4]). The time-like twistor lifts of the above surfaces are degenerate.

Suppose that N is Lorentzian. Then analogous discussions and results are also valid for space-like or time-like surfaces in N. In this case, we need the decomposition of the complexification of the two-fold exterior power of the pull-back bundle on a space-like or time-like surface, and the Levi-Civita connection of N gives connections of the two subbundles ([7]). The complex twistor spaces associated with the pull-back bundle are fiber bundles over the surface in the two subbundles. The complex twistor lifts of the surface are sections of the complex twistor spaces. We obtain a characterization of a space-like surface in N such that the covariant derivative of a complex twistor lift by $\partial/\partial w$ or $\partial/\partial \overline{w}$ for every local complex coordinate w of the surface vanishes ([7]). For a space-like surface with zero mean curvature vector, this property just means that a light-like normal vector field is parallel (refer to [1], [3], [8] for characterizations of such a surface). Moreover, if such a surface has no totally geodesic points, then it is strictly isotropic in the sense of [3] ([7]). We obtain characterizations of a time-like surface in N such that a complex twistor lift is horizontal along integral curves of a light-like one-dimensional distribution, and in particular, such a surface has zero mean curvature vector so that the shape operator of any normal vector field is zero or light-like ([7]). Refer to [8] for characterizations of such a surface based on [3]. The complex twistor lifts of the above surfaces are degenerate.

References

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