The Structure of D-homoderivations in Lie Algebra

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Lie algebras and their derivations are central to many areas of mathematics and theoretical physics [1]. The study of derivations, defined as linear maps that satisfy the Leibniz rule with respect to the Lie bracket, has expanded to include generalized derivations, quasi-derivations, centroids, and central derivations [2]. These generalizations offer a deeper understanding of the symmetry and automorphism properties of Lie algebras. Recent research has advanced the understanding of derivation-like structures under various algebraic constructions. Benkovič and Eremita [3] investigated conditions under which the decomposition $QDer(L) = Der(L) \oplus C(L)$ holds for quasi-derivations in current Lie algebras, particularly in the tensor product $L \otimes A$.

The study of derivation was initiated during the 1950s and 1960s. Strictly, derivations of rings got a tremendous development in 1957. Based on the fundamental definition of derivation $\zeta\colon \mathcal{G} \to \mathcal{G}$ satisfies $\zeta(xy) = \zeta(x)y + x\zeta(y)$ where $x, y \in \mathcal{G}$. When the year 2000 came, a classical definition concerning of homoderivation was delivered, where an additive mapping ζ is a homoderivation concerning a ring $\mathcal{G}$. More precisely, $\zeta\colon \mathcal{G} \to \mathcal{G}$ satisfies $\zeta(xy) = \zeta(x)\zeta(y) + \zeta(x)y + x\zeta(y)$ where x and y in $\mathcal{G}$. Lastly, in 2025, Mehsin Jabel Atteya via [4] contributed some promising results concerning the behavior of homogeneralized (σ, τ)-derivations and homogeneous (μ, ξ)-homoderivations on associative rings.

This paper presents explicit structural results concerning D-homoderivations in Lie algebras over arbitrary fields. It is established that the set of D-homoderivations forms a Lie algebra, which decomposes as the sum of homoderivations and centroids, intersecting precisely at the space of central homoderivations.

Definition 1. A linear mapping $\zeta\colon L \to L$ for any Lie algebra L is a D-homoderivations Lie algebra for which there exists a homoderivation D such that

$$\zeta([x, y]) = [\zeta(x), D(y)] + [\zeta(x), y] + [x, D(y)]$$

where $x, y \in L$. We denote by $Der_{HD}(L)$ for a D-homoderivations Lie algebra.

Theorem 2. *The $Der_{HD}(L)$ is a Lie subalgebra of $gl(L)$.*

Due to $Der_{HD}(L)$ forms a Lie algebra, it is clear that $Der(L)$ is a Lie subalgebra of $Der_{HD}(L)$. We now establish a decomposition of D-homoderivations into ordinary derivations and centroids.

Theorem 3. *Let L be a Lie algebra. Then, $Der_{HD}(L) = Der_{HD}(L) + C(L)$.*

References

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