On average elements of weighted sets

Vitaliy Bondarenko
Institute of Mathematics of NAS of Ukraine
E-mail: vitalij.bond@gmail.com

In mathematics, means of numerical collections represent the values that are the most typical in some sense. There are three classical means, namely the arithmetic mean (AM), the geometric mean (GM) and the harmonic mean (HM), and also the Lehmer and Hölder means, etc.

Now we proceed to the main definitions that introduce a new notion of average elements for finite weighted sets. For clarity, we consider the case of the arithmetic average, noting that other ones are treated similarly.

A *weighted set* is a pair (X, w), where X is a set and w is a mapping from X into the set $\mathbb{R}^+$ of nonnegative real numbers; $w(x)$ is called the *weight* of x.

Let (X, w) be a finite weighted set with $X = \{x_1, \dots, x_n\}$, and let x_w be the arithmetic mean of the numbers $w(x_1), \dots, w(x_n)$. Fix an element $x_i \in X$ such that the modulus of the difference $w(x_i) - x_w$ is minimal. In this case, we say that x_i is an *arithmetic average element* of the weighted set (X, w). Note that there may be several such elements.

We identify the singletons with the elements themselves. If the indicated difference is positive (resp. negative), the set of all average elements is called *right* (resp. *left*). Finally, if this difference is equal to 0, the set is called *exact*.

In this paper, we consider weighted sets of a special form when the sets are posets and the weights relate to the transitivity coefficients.

Let S be a finite poset and $S^2_{\prec} := \{(x, y) \mid x, y \in S,\ x \prec y\}$. Elements x and y are called *neighboring* if $(x, y) \in S^2_{\prec}$ and there is no z satisfying $x \prec z \prec y$. Let us denote by $n_w = n_w(S)$ the cardinality of the set $S^2_{\prec}$, and by $n_e = n_e(S)$ the number of pairs (x, y) of neighboring elements of S. The ratio $k_t = k_t(S)$ of the numbers $n_w - n_e$ and n_w is called the *coefficient of transitivity* of S ($k_t := 0$ for $n_w = 0$) [6].

In the language of the Hasse diagram $H(P)$ of S, n_e is equal to the number of all its edges, and n_w to the number of all paths directed bottom-up, up to parallelism (i.e., up to paths with the same start and the same terminal vertices).

We call a pair (S, k_t) a *weighted poset respectively the transitivity coefficient.* And an arbitrary class $\mathcal{C}$ of finite posets can be considered as the weighted class $(\mathcal{C}, k_t)$.

As an example, we consider the class of positive posets.

A finite poset $S \not\ni 0$ is called *positive* if so is its *Tits quadratic form* $q_S(z) : \mathbb{Z}^{\{0\} \cup S} \to \mathbb{Z}$, given by the following equality (see [7]):

$$q_S(z) = z_0^2 + \sum_{i \in S} z_i^2 + \sum_{i \prec j,\, i,j \in S} z_i z_j - z_0 \sum_{i \in S} z_i.$$

A positive poset S is called *serial* if there is an infinite increasing sequence $S \subset S^{(1)} \subset S^{(2)} \subset \dots$ with positive terms, and *nonserial* otherwise.

Serial and nonserial positive posets were first classified in [4, 3] (see also the more accessible paper [5]). The result was obtained by the minimax equivalence method [1] (for more detail, see [2]).

Now we proceed to the main theorems.

A poset T is called *dual* to a poset S and is denoted by S^{op} if it has the same elements and the inverse partial order relation. T is called *anti-isomorphic* to S if it is isomorphic to S^{op}.

We study only classes of posets closed under both isomorphism and duality (anti-isomorphism). If X is such a class, we consider its elements up to isomorphism and up to both isomorphism and duality. In the first case, the set of the corresponding classes is denoted by $[X]$, and in the second case by $[X]_{op}$. A complete system of representatives of $[X]$ (resp. $[X]_{op}$) is denoted by $[X]^0$ (resp. $[X]^0_{op}$).

Theorem 1. *Let $\mathcal{P}_0$ denote the class of the minimal nonserial positive posets. Then the following holds:*

(a) *there exists exactly one arithmetic average element of $\big([\mathcal{P}_0]^0_{op}, k_t\big)$, namely the one which is nonconnected and non-self-dual;*

(b) *there exist exactly two arithmetic average elements of $\big([\mathcal{P}_0]^0, k_t\big)$, namely the two mutually dual elements which are nonconnected and non-self-dual;*

(c) the set of average elements is left in case (a) and right in case (b).

Theorem 2. *Let $\mathcal{P}_0$ be as in Theorem 1. Then the sets of geometric average elements of $\big([\mathcal{P}_0]^0_{op}, k_t\big)$ and $\big([\mathcal{P}_0]^0, k_t\big)$ consist of all elements whose height is less than two. They contain, respectively, three and four elements, and both are exact.*

References

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