Linearization of Hamilton–Jacobi Equations via Curved Manifolds of Schwartz Waves and Schrödinger Dynamics

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This contribution proposes a geometric and Schwartz-linear reinterpretation of the relationship between Hamilton–Jacobi theory and Schrödinger quantum dynamics. The central idea is that suitable Hamilton principal functions generate, through their quantum exponentials, curved manifolds of Schwartz waves whose Schwartz-superpositional completion produces exact quantum dynamics. Starting from admissible Hamilton–Jacobi families $S = (S_\lambda)_\lambda$, one considers the associated principal exponential family $v_\lambda = \exp(iS_\lambda/\hbar)$. Under suitable spectral admissibility assumptions, these exponentials form continuous or discrete Schwartz spectral bases for quantum Hamiltonians. The Schrödinger equation then appears as the exact linear Schwartz-superpositional extension of Hamilton–Jacobi spectral geometry. The framework simultaneously covers: free particles; relativistic de Broglie waves; particles on compact manifolds; standing-wave systems; harmonic oscillators; continuous spectra; discrete spectra; and finite spectral blocks. The theory also leads to a generalized Schwartz–von Neumann extension principle, according to which classical observables acting on certainty spectral germs extend uniquely to continuous Schwartz-linear operators on the corresponding quantum distribution spaces. The resulting picture suggests that quantum dynamics may be interpreted as the Schwartz-superpositional completion of an underlying geometry of classical certainty germs generated by admissible Hamilton–Jacobi structures.

One of the oldest conceptual problems in mathematical physics concerns the relationship between classical Hamilton–Jacobi theory and quantum wave dynamics. Historically, the Hamilton–Jacobi equation provided a geometric and variational description of classical mechanics through action functions and characteristic trajectories, while Schrödinger quantum mechanics introduced linear wave evolution in Hilbert spaces. The de Broglie relation $\psi = \exp(iS/\hbar)$ already suggested a profound connection between these two theories. However, in standard formulations, the relationship often remains local, semiclassical, asymptotic, or purely heuristic. The present work develops a different perspective based on Schwartz linear algebra, tempered distributions, and spectral superposition theory. The central observation is that Hamilton principal functions may generate exact Schwartz spectral families whose superpositions solve the Schrödinger equation globally and rigorously. The geometric mechanism may be summarized as follows:

$$\text{HJ principal functions} \to \text{principal exponentials} \to \\ \to \text{Schwartz spectral families} \to \text{quantum dynamics.}$$

In the canonical relativistic case, one starts from Minkowski covectors $p \in M_4^*$ and their associated affine Hamilton principal functions $S_p(x) = \langle p, x\rangle$. Their exponentials are precisely the relativistic de Broglie waves $\eta_p(x) = \exp\left(\frac{i}{\hbar}\langle p, x\rangle\right)$. The flat affine geometry of Minkowski bras thus generates a curved manifold of principal waves inside tempered-distribution state spaces. The Schrödinger equation appears after imposing the relativistic Hamilton–Jacobi compatibility relation $-\partial_t S_p = E(p)$. The resulting spectral geometry is then extended by Schwartz superposition. This viewpoint naturally generalizes: from continuous to discrete spectra; from free particles to bounded systems; from single waves to finite spectral blocks; and from affine action geometries to curved and logarithmic action structures. The present contribution focuses especially on the geometric meaning of these constructions and on the emerging concept of certainty spectral germs.

The theory is developed inside the framework of Schwartz linear algebra and tempered distribution spaces. Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz test-function space and $\mathcal{S}'(\mathbb{R}^n)$ its tempered dual. A Schwartz family $v = (v_\lambda)_{\lambda\in\Lambda}$ is a family of tempered distributions such that the coefficient transform $\phi \mapsto \langle v, \phi\rangle$ has suitable Schwartz regularity properties. If the family forms a Schwartz basis, every state admits a superpositional representation $u = \int_\Lambda a\, v$, where a is a suitable coefficient distribution. This provides the appropriate framework for generalized spectral expansions. The theory introduces admissible Hamilton–Jacobi families: $S = (S_\lambda)_\lambda$. Their principal exponentials are $v_\lambda = \exp(iS_\lambda/\hbar)$. A family S is H-spectrally admissible if: the exponential v forms a Schwartz basis; the generated space is invariant under the quantum Hamiltonian $\hat{H}$; each exponential is an eigenvector: $\widehat{H}v_\lambda = E(\lambda)v_\lambda$. Under these assumptions, every admissible superposition $\psi = \int a\, v$ satisfies the Schrödinger equation $i\hbar\partial_t\psi = \widehat{H}\psi$. The framework also yields a generalized Schwartz–von Neumann extension principle. Given a classical observable $f : \Lambda \to \mathbb{R}$, its spectral extension is the unique continuous Schwartz-linear operator satisfying $\widehat{f}(v_\lambda) = f(\lambda)v_\lambda$. This extends the ordinary diagonal spectral construction to generalized continuous and distributional spectral geometries. A central notion emerging from the theory is the concept of certainty spectral germ. The certainty germ is the elementary spectral object generating the quantum dynamics. Its geometry depends on the physical system under consideration. Examples include: four-momenta for relativistic free particles; quantized angular momenta for particles on a circle; paired opposite momenta for standing-wave systems; quantized action shells for harmonic oscillators. Thus, quantum state spaces appear as Schwartz-superpositional completions of classical certainty-germ geometries.

The theory shows that Schrödinger quantum dynamics may be interpreted as the linearization of Hamilton–Jacobi spectral geometry through principal exponentials. More precisely, nonlinear Hamilton–Jacobi structures generate curved manifolds of Schwartz waves inside distribution spaces, while the Schrödinger equation governs their linear Schwartz-superpositional evolution. Several different spectral geometries are shown to fit into the same mechanism.

Continuous spectral geometry. For free relativistic particles, the certainty germs are Minkowski covectors $p \in M_4^*$. The associated Hamilton principal functions are $S_p(x) = \langle p, x\rangle$, and the principal waves are $\eta_p(x) = \exp(iS_p(x)/\hbar)$. The relativistic Hamilton–Jacobi relation reduces the admissible spectral geometry to the mass shell $\langle p, p\rangle = -m_0^2c^2$. The Schrödinger equation then becomes the exact superpositional extension of the corresponding principal-wave geometry.

Compact spectral geometry. For particles on a ring, the certainty germs are quantized angular momenta $p_n = \hbar n$. The associated actions are $S_n(t, \theta) = \hbar n\theta - E_n t$. Their exponentials $\exp(iS_n/\hbar)$ form the Fourier basis on the circle. The discreteness arises from topological single-valuedness of the principal exponentials.

Boundary spectral geometry. For the particle in a box, the elementary certainty structure is no longer a single momentum but a paired spectral block: $(p, -p)$. The physical standing waves arise as admissible antisymmetric combinations of the two opposite traveling principal exponentials. This shows that admissible Hamilton–Jacobi structures may possess internal finite-dimensional spectral geometry before the final quantum basis is selected.

The theory suggests the following geometric classification.

System	Certainty Germ	Spectral Type	Action Geometry
Free particle	four-momentum	continuous	real affine
Particle on a ring	angular momentum	discrete	real periodic
Particle in a box	paired momenta	discrete block	paired affine

These examples show that the same Schwartz-superpositional mechanism survives under: translational geometry; compact topology; boundary constraints; standing-wave selection; localization; and nodal spectral geometry. The framework therefore extends far beyond ordinary Fourier analysis. The free particle corresponds only to the flat translational realization of a much more general Hamilton–Jacobi spectral mechanism.

References

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