

Simplicity of Lie algebras of vector fields

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Let K be an algebraically closed field of characteristic zero, let $A = K[x_1, \dots, x_n]$ be the polynomial ring in n variables, and let $R = K(x_1, \dots, x_n)$ be the field of rational functions. A K -derivation on the ring A is a K -linear map $D : A \rightarrow A$ such that

$$D(fg) = D(f)g + fD(g)$$

for all $f, g \in A$. Every K -derivation can be uniquely written in the form

$$D = f_1 \frac{\partial}{\partial x_1} + \dots + f_n \frac{\partial}{\partial x_n}, \quad f_i = f_i(x_1, \dots, x_n) \in A,$$

where $\frac{\partial}{\partial x_i}$ are the usual partial derivatives. The K -vector space of all derivations of the ring A is a Lie algebra over K with respect to the Lie bracket $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$. We denote this Lie algebra by $W_n(K)$. It follows from the above that $W_n(K)$ is a free A -module with the standard basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. The Lie algebra $W_n(K)$ is of great interest since, from the geometric point of view, $W_n(K)$ consists of all polynomial vector fields on K^n . The structure of subalgebras of the Lie algebra $W_n(K)$, $n \geq 2$ has been studied by many authors, but many open problems remain in this direction; see, for example, [5, 1]. The subalgebras of the Lie algebra $W_1(K)$ are described in [2]. In [4], it was shown that there are close connections between ideals of the polynomial ring A and subalgebras of the Lie algebra $W_n(K)$. Especially interesting are subalgebras of the Lie algebra $W_n(K)$ which are simultaneously submodules of the free A -module $W_n(K)$. We call such subalgebras polynomial subalgebras (see also [3]). One of the main results of this paper states that every Lie algebra $L \subseteq W_n(K)$ having no nonzero proper L -invariant ideals is a simple Lie algebra. We also point out another source of simple subalgebras, namely maximal subalgebras of rank $k < n$ of the Lie algebra $W_n(K)$. Here the rank $\text{rk } L$ of a subalgebra $L \subseteq W_n(K)$ is, as usual, the dimension $\dim_R RL$. Note that maximal subalgebras of rank $< n$ are automatically polynomial subalgebras.

Using results of the paper [4] we obtain the following result.

Lemma 1. *Let $X = V(I)$, $X \subset K^n$ be an affine algebraic variety with the defining ideal I , and let $p \in K^n \setminus X$ be a point. Then, for an arbitrary vector $\vec{u} = (u_1, \dots, u_n) \in K^n$, there exists a derivation*

$$D = \sum_{i=1}^n f_i(x_1, \dots, x_n) \frac{\partial}{\partial x_i} \in W_n$$

such that:

- (1) $D(I) \subseteq I$;
- (2) $v_D(p) = (u_1, \dots, u_n)$, where v_D is the vector field on K^n corresponding to D .

Theorem 2. *Let M be a maximal subalgebra of the Lie algebra W_n with $\text{rk } M = k < n$. Then*

- (1) *the only M -invariant ideals of the polynomial ring A are A and 0 .*
- (2) *M is a simple subalgebra of the Lie algebra W_n .*

The situation for maximal polynomial subalgebras of rank n is opposite as the next statement shows.

Theorem 3. *Let M be a polynomial maximal subalgebra of rank n from W_n . Then*

- (1) $M = \{D \in W_n \mid D(I) \subseteq I\}$ for an ideal I of the ring A .
 (2) The subalgebra I^2W_n is a nonzero proper ideal of M , so M is a non-simple Lie algebra.

We use here properties of modules of maximal rank over the polynomial ring A . If M is such a module then $I = \{f \in A \mid fW_n(K) \subseteq M\}$ is a proper nonzero ideal of the ring A .

REFERENCES

- [1] V. V. Bavula. The groups of automorphisms of the Lie algebras of polynomial vector fields with zero or constant divergence. *Comm. Algebra*, 45(3):1114–1133, 2017. doi:[10.1080/00927872.2016.1175596](https://doi.org/10.1080/00927872.2016.1175596).
- [2] Jason Bell and Lucas Buzaglo. Maximal dimensional subalgebras of general Cartan-type Lie algebras. *Bull. Lond. Math. Soc.*, 57(2):605–624, 2025. doi:[10.1112/blms.13216](https://doi.org/10.1112/blms.13216).
- [3] V. M. Buchstaber and D. V. Leykin. Polynomial Lie algebras. *Funct. Anal. Appl.*, 36(4):267–280, 2002. doi:[10.1023/A:1021757609372](https://doi.org/10.1023/A:1021757609372).
- [4] D. A. Jordan. On the ideals of a Lie algebra of derivations. *J. London Math. Soc. (2)*, 33(1):33–39, 1986. doi:[10.1112/jlms/s2-33.1.33](https://doi.org/10.1112/jlms/s2-33.1.33).
- [5] Ie. O. Makedonskyi and A. P. Petravchuk. On nilpotent and solvable Lie algebras of derivations. *J. Algebra*, 401:245–257, 2014. doi:[10.1016/j.jalgebra.2013.10.030](https://doi.org/10.1016/j.jalgebra.2013.10.030).