Conformal mappings and invariance of the Einstein equations

Yevhen Cherevko
Department of Artificial Intelligence and Mathematical Modeling, National University "Odesa Law Academy" 23, Fontanska str., 65009, Odesa, Ukraine
E-mail: cherevko@usa.com

Olena Chepurna
Department of Cybersecurity, National University "Odesa Law Academy" 23, Fontanska str., 65009, Odesa, Ukraine
E-mail: chepurna@onua.edu.ua

Yevheniia Kuleshova
Department of Algebra and Geometry, Faculty of Science, Palacký University Olomouc Křížkovského 511/8, CZ-771 47 Olomouc, Czech Republic
E-mail: yevheniia.kuleshova01@upol.cz

Let ψ be a spinor field in an n-dimensional space-time $(V^{1,n-1}, g)$. The Belinfante–Rosenfeld stress-energy tensor of the spinor field is calculated by the formula [1]

$$T_{jk} = \frac{i}{2}\big(\overline{\psi}\gamma_{(j}\nabla_{k)}\psi - (\nabla_{(j}\overline{\psi})\gamma_{k)}\psi\big). \tag{1}$$

It should satisfy the Einstein equations [1]

$$R_{ij} - \frac{1}{2}Rg_{ij} = -8\pi G T_{ij}, \tag{2}$$

where g_{ij}, $R_{ij} = R^{\alpha}_{ij\alpha}$ and $R = R_{ij}g^{ij}$ are the metric, the Ricci tensor and the scalar curvature respectively of a space-time $(V^{1,n-1}, g)$. Obviously, R^h_{ijk} is the curvature tensor of $(V^{1,n-1}, g)$.

Let us consider a conformal mapping $f : (V^{1,n-1}, g) \to (\tilde{V}^{1,n-1}, \tilde{g})$, i. e. a diffeomorphism between the manifolds whose metrics are related as

$$\tilde{g}_{ij}(x) = e^{2\varphi(x)}g_{ij}(x),$$

where $\varphi(x)$ is a function of the x's. The following relations hold [3, 4]

$$\begin{aligned} \tilde{R}^h_{ijk} &= R^h_{ijk} + \delta^h_k\varphi_{ij} - \delta^h_j\varphi_{ik} + g^{hl}(\varphi_{lk}g_{ij} - \varphi_{lj}g_{ik}) + (\delta^h_k g_{ij} - \delta^h_j g_{ik})\Delta_1\varphi, \\ \tilde{R}_{ij} &= R_{ij} + (n-2)\varphi_{ij} + (\Delta_2\varphi + (n-2)\Delta_1\varphi)g_{ij}, \\ \tilde{R} &= e^{-2\varphi}(R + 2(n-1)\Delta_2\varphi + (n-1)(n-2)\Delta_1\varphi). \end{aligned} \tag{3}$$

Here $\varphi_i = \partial_i\varphi$, $\Delta_1\varphi = \varphi_i\varphi_j g^{ij}$, $\varphi_{ij} = \nabla_j\varphi_i - \varphi_i\varphi_j$, $\Delta_2\varphi = \nabla_j\varphi_i g^{ij}$. We know that invariance of the Dirac equations requires that a spinor field should be transformed according to the formula [1]

$$\psi' = e^{s\varphi(x)}\psi.$$

The number $s = \frac{1-n}{2}$ is referred to as the conformal weight of a spinor field ψ.

Then the Belinfante–Rosenfeld stress-energy tensor (1) of a spinor field is transformed within a conformal mapping according to the formula

$$\tilde{T}_{jk} = e^{(2s+1)\varphi}T_{jk}. \tag{4}$$

Note that for a massless spinor field [2, 5]

$$\tilde{T} = \tilde{T}_{jk}\tilde{g}^{jk} = T = T_{jk}g^{jk} = 0.$$

It follows from (2), (3) that the function φ must satisfy the system (massless case)

$$\nabla_j\varphi_i = \varphi_i\varphi_j - \frac{1}{2}g_{ij}\Delta_1\varphi - \frac{8\pi G}{n-2}\left(\tilde{T}_{ij} - T_{ij}\right),$$

or, taking into account (4),

$$\nabla_j\varphi_i = \varphi_i\varphi_j - \frac{1}{2}g_{ij}\Delta_1\varphi - \frac{8\pi G T_{ij}}{n-2}\left(e^{(2s+1)\varphi} - 1\right). \tag{5}$$

Conditions of integrability of (5) are [6]

$$\varphi_\alpha R^{\alpha}_{ijk} + \frac{8\pi G}{n-2}\big((\nabla_k T_{ij} - \nabla_j T_{ik})\left(e^{(2s+1)\varphi} - 1\right) + (\varphi_k T_{ij} - \varphi_j T_{ik})(2s+1)e^{(2s+1)\varphi}\big).$$

References

[1] N. D. Birrell and P. C. W. Davies. *Quantum Fields in Curved Space*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1982. doi:10.1017/CBO9780511622632.

[2] Peter Collas and David Klein. *The Dirac Equation in Curved Spacetime: A Guide for Calculations*. SpringerBriefs in Physics. Springer, Cham, 2019. doi:10.1007/978-3-030-14825-6.

[3] Luther Pfahler Eisenhart. *Riemannian Geometry*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Reprint of the 1926 original.

[4] Josef Mikeš et al. *Differential Geometry of Special Mappings*. Palacký University Press, Olomouc, 2nd edition, 2019. doi:10.5507/prf.19.24455358.

[5] Yoshiaki Tanii. *Introduction to Supergravity*, volume 1 of *SpringerBriefs in Mathematical Physics*. Springer, Tokyo, 2014. doi:10.1007/978-4-431-54828-7.

[6] Kentaro Yano. *The Theory of Lie Derivatives and Its Applications*, volume 3 of *Bibliotheca Mathematica*. North-Holland Publishing Co., Amsterdam; P. Noordhoff Ltd., Groningen; Interscience Publishers Inc., New York, 1957.