Evolutionary inner radii estimates for systems of domains with given parameters

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This work investigates the problem of estimating functionals defined on systems of mutually non-overlapping domains in the complex plane. In particular, we obtain upper bounds for products of the inner radii of n such domains with respect to systems of fixed points. The main result establishes the dependence of these upper bounds on a real parameter γ for the entire range of its values $\gamma \in (0, n]$.

Let $r(B, a)$ be the inner radius of a domain $B \subset \overline{\mathbb{C}}$ with respect to a point $a \in B$. The inner radius is a generalization of the conformal radius to the case of multiply connected domains. The inner radius of the domain B is related to the generalized Green function $g_B(z, a)$ of the domain B by the following relations:

$$g_B(z,a) = \ln \frac{1}{|z-a|} + \ln r(B,a) + o(1), \quad z \to a,$$

$$g_B(z,\infty) = \ln |z| + \ln r(B,\infty) + o(1), \quad z \to \infty.$$

This work is devoted to obtaining evolutionary-type inequalities for functionals of the following form for all values of the parameter $\gamma \in (0,\, n]$:

$$I_n(\gamma) = r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

$$Y_n(\gamma) = r^{\gamma}\left(B_\infty, \infty\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

$$J_n(\gamma) = \left(r\left(B_0, 0\right) r\left(B_\infty, \infty\right)\right)^{\gamma} \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

$$I_n(\gamma_1, \ldots, \gamma_n) = \prod_{k=1}^{n} r^{\gamma_k}(B_k, a_k),$$

where $n \in \mathbb{N}$, $\{a_k\}_{k=1}^{n}$ is an arbitrary fixed system of points in the complex plane $\mathbb{C}\backslash\{0\}$; B_0, B_∞, $\{B_k\}_{k=1}^{n}$ is an arbitrary system of pairwise non-overlapping domains such that $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, and $a_k \in B_k \subset \overline{\mathbb{C}}$ for $k = \overline{1,n}$.

These functionals have been studied in numerous papers under various conditions (see, e.g., [1–5]), but currently, there are no methods for obtaining their exact maximums for all values of the parameter $\gamma \in (0,\, n]$. In particular, in [1], dynamic estimates were proven for the functionals $I_n(\gamma)$, $Y_n(\gamma)$, and $J_n(\gamma)$, expressed in terms of their initial values $I_n(0)$, $Y_n(0)$, and $J_n(0)$, respectively, and a generalization of M.A. Lavrentiev's result on the maximum product of conformal radii of two non-overlapping simply connected domains was obtained. In the present work, we obtain evolutionary estimates for the products of inner radii of pairwise non-overlapping multi-connected domains with respect to fixed points in the complex plane.

Theorem 1. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\varepsilon > 0$, $\gamma \in (0,\, n]$, and $\tau \in (0,\, \gamma)$. Then, for any fixed system of distinct points $\{a_k\}_{k=1}^{n} \in \mathbb{C}\backslash\{0\}$ and any collection of pairwise non-overlapping domains B_k, $k = \overline{0,n}$,* *such that* $a_k \in B_k \subset \overline{\mathbb{C}}$, $k=\overline{0,n}$, $a_0=0$, *and* $r(B_k,a_k) \geqslant \varepsilon$ *for* $k=\overline{1,n}$, *the following inequality holds:*

$$I_n(\gamma) \leqslant n^{-\frac{\gamma-\tau}{2}} \varepsilon^{\tau-\gamma} I_n(\tau) \left(\prod_{k=1}^{n} |a_k|\right)^{\frac{2(\gamma-\tau)}{n}}.$$

Theorem 2. *Let* $n \in \mathbb{N}$, $n \geqslant 2$, $\varepsilon > 0$, $\gamma \in (0, n]$, *and* $\tau \in (0, \gamma)$. *Then, for any fixed system of distinct points* $\{a_k\}_{k=1}^{n} \in \mathbb{C}$ *and any collection of pairwise non-overlapping domains* B_∞, B_k, *such that* $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k=\overline{1,n}$, *and provided that* $r(B_k,a_k) \geqslant \varepsilon$ *for all* $k=\overline{1,n}$, *the following inequality holds:*

$$Y_n(\gamma) \leqslant n^{-\frac{\gamma-\tau}{2}} \varepsilon^{\tau-\gamma} Y_n(\tau).$$

Theorem 3. *Let* $n \in \mathbb{N}$, $n \geqslant 2$, $\varepsilon > 0$, $\gamma \in (0,n]$, *and* $\tau \in (0, \gamma)$. *Then, for any fixed system of distinct points* $\{a_k\}_{k=1}^{n} \in \mathbb{C}\backslash\{0\}$ *and any collection of pairwise non-overlapping domains* B_0, B_∞, B_k, *such that* $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, *and* $a_k \in B_k \subset \overline{\mathbb{C}}$ *for* $k=\overline{1,n}$, *provided that* $r(B_k,a_k) \geqslant \varepsilon$ *for all* $k=\overline{1,n}$, *the following inequalities hold:*

$$J_n(\gamma) \leqslant \begin{cases} (n+1)^{-(\gamma-\tau)\frac{n+1}{n+2}} \varepsilon^{-\frac{2n(\gamma-\tau)}{n+2}} J_n(\tau) \prod\limits_{k=1}^{n} |a_k|^{\frac{2(\gamma-\tau)}{n+2}}, & \text{if } \gamma \in \left(0, \frac{n+2}{2}\right]; \\ (n+1)^{-\frac{n+1}{2}} \prod\limits_{k=1}^{n} |a_k|, & \text{if } \gamma \in \left(\frac{n+2}{2}, n\right]. \end{cases}$$

Theorem 4. *Let* $n \in \mathbb{N}$, $n \geqslant 3$, *and* γ_k, θ_k, $k=\overline{1,n}$, *be some positive real numbers such that* $\gamma_k \geqslant \theta_k$ *for* $k=\overline{1,n}$. *Then, for any fixed system of distinct points* $\{a_k\}_{k=1}^{n} \in \mathbb{C}$ *and any collection of domains* $B_k \subset \overline{\mathbb{C}}$, $k=\overline{1,n}$, *such that* $a_k \in B_k$, $k=\overline{1,n}$, $B_i \cap B_j = \emptyset$ *for* $i \neq j$, *and provided that* $r(B_k,a_k) \geqslant \varepsilon$ *for* $k=\overline{1,n}$ *and some* $\varepsilon > 0$, *the following inequality holds:*

$$I_n(\gamma_1,\ldots,\gamma_n) \leqslant \varepsilon^{-\sum\limits_{k=1}^{n}(\gamma_k-\theta_k)} (n-1)^{-\frac{1}{2}\sum\limits_{k=1}^{n}(\gamma_k-\theta_k)} \prod_{i,j=1, i<j}^{n} |a_j-a_i|^{\frac{2}{n-1}(\gamma_i+\gamma_j-\theta_i-\theta_j)} \prod_{k=1}^{n} r^{\theta_k}(B_k,a_k).$$

The above Theorems allow us to obtain certain estimates of evolutionary type. These estimates can be useful, for example, when we know the exact estimates for a certain fixed configuration of points a_k and a certain set of exponents θ_k, but we need to find estimates for the same configuration of points a_k but for a different set of exponents α_k. The estimates obtained in this manner can be more precise than those obtained directly.

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References

[1] Bakhtin A.K., Denega I.V. Generalized M.A. Lavrentiev's inequality. *J. Math. Sci.*, 262(2) : 138–153, 2022.

[2] Denega I., Zabolotnyi Y. Some extremal problems on the Riemannian sphere. *Carpathian Math. Publ.*, 16(2) : 593–605, 2024.

[3] Denega I.V., Zabolotnyi Ya.V. Application of upper estimates for products of inner radii to distortion theorems for univalent functions. *Matematychni Studii*, 60(2) : 138–144, 2023.

[4] Denega I., Zabolotnyi Y. Using conformal radii of the open unit disk sectors in distortion theorems for univalent functions. *Complex Analysis and its Synergies*, 11 : article number 8, 2025.

[5] Denega I., Zabolotnyi Y. Estimates for sums of logarithmic potentials associated with the Green function. *Researches in Mathematics*, 33(3) : 14–22, 2025.