Singularities of real nodal curves

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It is a joint work with Igor Burban. For details see [2].

Let $(X, \mathcal{A})$ be a reduced non-commutative curve [3], that is X is an algebraic curve over a field $\Bbbk$ and $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-algebras coherent and torsion free as $\mathcal{O}_X$-module and without nil-ideals. Without loss of generality, we can suppose that this non-commutative curve is *central*, that is $\mathcal{O}_x$ coincides with the center of $\mathcal{A}_x$ for every point $x \in X$. In "usual" (commutative) case it means that $\mathcal{A} = \mathcal{O}_X$.

We denote by $\mathcal{K}$ the sheaf of rational functions on X, set $\mathcal{KA} = \mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{A}$. Note that for every $x \in X$ the localization $\mathcal{A}_x$ is an $\mathcal{O}_x$-*order* in $\mathcal{KA}$. We say that a point $x \in X$ is *singular for* $\mathcal{A}$ if $\mathcal{A}_x$ is not a maximal order. We call the completion $\hat{\mathcal{A}}_x$ of this localization a *singularity* of the non-commutative curve $(X, \mathcal{A})$. This singularity is called *nodal* if

(1) $\operatorname{End}_{\mathcal{A}_x} \mathcal{J}_x = \mathcal{H}_x$ is hereditary, where $\mathcal{J}_x = \operatorname{rad} \mathcal{A}_x$,
(2) $\operatorname{rad} \mathcal{H}_x = \mathcal{J}_x$,
(3) $\ell_{\mathcal{A}_x}(\mathcal{H}_x \otimes_{\mathcal{A}_x} U) \leq 2$ for every simple $\mathcal{A}_x$-module U.

(Note that these conditions hold for $\mathcal{A}_x$ if and only if they hold for $\hat{\mathcal{A}}_x$).

It is known [1] that up to Morita equivalence the non-commutative curve $(X, \mathcal{A})$ is defined by the $\mathcal{K}$-algebra $\mathcal{KA}$ and the rings $\mathcal{A}_x$ for the points x singular for $\mathcal{A}$.

If the basic field $\Bbbk$ is algebraically closed, the possible singularities were classified by Voloshyn [4]. We describe a general procedure that allows to construct nodal singularities in general case and classify them if the basic field is the field of real numbers.

References

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[4] D. Voloshyn. Structure of nodal algebras. Ukrainian Math. J. 63(7), 1013–1022. (2011).