On a three-point extension of Chatterjea-type mappings

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In 1972, S. Chatterjea [3] proved a fixed point theorem for mappings that may be discontinuous on their domain: let $T\colon X \to X$ be a mapping on a complete metric space (X, d) with

$$d(Tx, Ty) \leqslant \gamma(d(x, Ty) + d(y, Tx))$$

where $0 \leqslant \gamma < \frac{1}{2}$ and $x, y \in X$. Then T has a unique fixed point. In [1], a class of generalized Chatterjea-type mappings with contraction coefficient $\lambda \in [0, \frac{1}{3})$ was introduced, and a fixed-point theorem for these mappings was proved. A little later C. M. Păcurar and O. Popescu [6] improved this result, see Theorem 2, extending the value of λ to $[0, \frac{1}{2})$.

Definition 1. Let (X, d) be a metric space with $|X| \geqslant 3$. We shall say that $T\colon X \to X$ is a *generalized Chatterjea type mapping* on X if there exists $\gamma \in [0, \frac{1}{2})$ such that the inequality

$$\begin{aligned} &d(Tx, Ty) + d(Ty, Tz) + d(Tx, Tz) \\ &\leqslant \gamma(d(x, Ty) + d(x, Tz) + d(y, Tx) + d(y, Tz) + d(z, Tx) + d(z, Ty)) \end{aligned}$$

holds for all three pairwise distinct points $x, y, z \in X$.

Independently, it was shown in [1, 6] that usual Chatterjea-type mappings are generalized Chatterjea-type mappings.

Let T be a mapping on the metric space X. A point $x \in X$ is called a *periodic point of period* n if $T^n(x) = x$. The least positive integer n for which $T^n(x) = x$ is called the prime period of x. In particular, the point x is of prime period 2 if $T^2(x) = x$ and $Tx \neq x$.

Theorem 2. *Let (X, d), $|X| \geqslant 3$, be a complete metric space and let the mapping $T\colon X \to X$ satisfy the following two conditions:*

(i) *T does not possess periodic points of prime period 2.*

(ii) *T is a generalized Chatterjea type mapping on X.*

Then T has a fixed point. The number of fixed points is at most two.

Recall that for a given metric space X, a point $x \in X$ is said to be an *accumulation point* of X if every open ball centered at x contains infinitely many points of X.

Proposition 3. *Let (X, d) be a metric space and let $T\colon X \to X$ be a generalized Chatterjea type mapping. If x is an accumulation point of X and T is continuous at x, then the inequality*

$$d(Tx, Ty) \leqslant \frac{\gamma}{1-\gamma}(2d(x, Ty) + d(y, Tx))$$

holds for all points $y \in X$.

Corollary 4. *Let (X, d) be a metric space, $T\colon X \to X$ be a continuous generalized Chatterjea type mapping with $\gamma \in [0, \frac{1}{4})$ and let all points of X be accumulation points. Then T is a Chatterjea type mapping.*

The mappings contracting perimeters of triangles and generalized Kannan type mappings were considered in [4] and [5], respectively. The following three propositions establish connections between these mappings and generalized Chatterjea type mappings.

Proposition 5. *Let (X, d) be a metric space with $|X| \geq 3$, and $T\colon X \to X$ be a mapping contracting perimeters of triangles with a constant $0 \leqslant \alpha < \frac{1}{3}$. Then T is a generalized Chatterjea type mapping.*

Proposition 6. *Let (X, d) be a metric space with $|X| \geq 3$, and $T\colon X \to X$ be a generalized Kannan mapping with a constant $0 \leqslant \lambda < \frac{1}{2}$. Then T is a generalized Chatterjea type mapping.*

Proposition 7. *Let (X, d) be a metric space with $|X| \geq 3$, and $T\colon X \to X$ be a generalized Chatterjea type mapping with a constant $0 \leqslant \gamma < \frac{1}{5}$. Then T is a generalized Kannan mapping.*

Proposition 8. *Generalized Chatterjea type mappings are continuous at fixed points.*

In the following two theorems, we exclude the completeness requirement of the metric space, compare with Theorem 2.

Theorem 9. *Let (X, d), $|X| \geqslant 3$, be a metric space and let the mapping $T\colon X \to X$ satisfy the following four conditions:*

(i) *T does not possess periodic points of prime period 2.*
(ii) *T is a generalized Chatterjea type mapping on X.*
(iii) *T is continuous at $x^* \in X$.*
(iv) *There exists a point $x_0 \in X$ such that the sequence of iterates $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$, has a subsequence x_{n_k}, converging to x^*.*

Then x^* is a fixed point of T. The number of fixed points is at most two.

Theorem 10. *Let (X, d), $|X| \geqslant 3$, be a metric space and let the mapping $T\colon X \to X$ be continuous. Suppose that*

(i) T does not possess periodic points of prime period 2.
(ii) T is a generalized Chatterjea type mapping on (M, d), where M is an everywhere dense subset of X.
(iii) There exists a point $x_0 \in X$ such that the sequence of iterates $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$, has a subsequence x_{n_k}, converging to x^*.

Then x^* is a fixed point of T. The number of fixed points is at most two.

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