

On partial preliminary group classification of some class of $(1+3)$ -dimensional Monge–Ampère equations. Three-dimensional Lie algebras

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The study of differential geometry, Kähler geometry and the Calabi conjecture, theoretical physics, optimal mass transportation, reflector and lens design in geometric optics, semi-geostrophic equations in meteorological modeling, elasticity, image processing and image registration, and optimal control in mathematical finance all lead, in various dimensions, to equations of Monge–Ampère type — reflecting their remarkable universality across mathematics and its applications. Numerous papers and monographs have been dedicated to the analysis of these classes of equations from various methodological perspectives.

We consider the following class of $(1+3)$ -dimensional Monge–Ampère equations:

$$\det(u_{\mu\nu}) = F(x_0, x_1, x_2, x_3, u, u_0, u_1, u_2, u_3),$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3) \in M(1, 3)$,

$$u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}, \quad u_\alpha \equiv \frac{\partial u}{\partial x_\alpha}, \quad \mu, \nu, \alpha = 0, 1, 2, 3.$$

Here $M(1, 3)$ is a four-dimensional Minkowski space, and F is an arbitrary real smooth function.

We continue using the classical Lie–Ovsiannikov approach for partial preliminary group classification of this class. In [10, 11] we have performed the classification using one- and two-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1, 4)$.

I plan to present some of the results obtained using three-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1, 4)$.

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