Exponential Tail Estimates for Lacunary Trigonometric Series

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We establish explicit non-asymptotic upper bounds for the tail probabilities of normalized lacunary sums of the form

$$\nu_N(x) = \frac{f_N(x)}{\sigma(N)}$$

where

$$f_N(x) := \sum_{k=1}^{N} c_k \sqrt{2}\cos(2\pi n_k x), \quad x \in [0,1],$$

$\{n_k\}$ is a strictly increasing sequence of integers satisfying the Hadamard gap condition:

$$n_{k+1} \geq q\, n_k, \quad q > 1, \quad \forall\, k \geq 1,$$

$\{c_k\}$ is a sequence of real numbers not square-summable, i.e.,

$$\sum_{k=1}^{\infty} c_k^2 = \infty,$$

$\sigma^2(N)$ is the variance of f_N defined by

$$\sigma^2(N) := \mathrm{Var}(f_N) = \mathbb{E}[f_N^2] - (\mathbb{E}[f_N])^2,$$

$\mathbb{E}[f_N]$ is the expectation given by

$$\mathbb{E}[f_N] = \int_0^1 f_N(x)\, dx.$$

The variance satisfies

$$\sigma^2(N) = \sum_{k=1}^{N} c_k^2.$$

Since the variance diverges, the normalized sums exhibit non-trivial behavior.

In particular, we have

$$\mathbb{E}[\nu_N] = 0, \qquad \mathrm{Var}(\nu_N) = 1.$$

Classical asymptotic results describe the behavior of ν_N as $N \to \infty$, while we provide explicit bounds for fixed N. More precisely, for fixed N, explicit exponential estimates are obtained for the tail probabilities

$$T[\nu_N](t) = \mathbb{P}\{x \in [0,1] : \nu_N(x) \geq t\}, \qquad t > 0,$$

in terms of the lacunarity parameter q and the normalizing factor $\sigma(N)$.

Furthermore, some examples are provided to illustrate the obtained bounds for different choices of coefficients c_k, highlighting the transition from subgaussian to stretched-exponential tail behavior.

This study is motivated by several applications, including Fourier series, signal processing, and the analysis of stochastic processes, where explicit non-asymptotic bounds for the tail probabilities of partial sums play an important role.

The results presented here are contained in [1].

References

[1] M.R. Formica, E. Ostrovsky, L. Sirota, Exponential Tail Estimates for Lacunary Trigonometric Series. *Axioms*, 15 (1), Article 5 (2026). DOI: 10.3390/axioms15010005