Algebraic analogue for Swan's local–global extension lemma

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The Serre–Swan correspondence provides a connection between algebra and geometry. It identifies vector bundles with finitely generated projective modules: in algebraic geometry, locally free sheaves of finite rank on an affine scheme correspond to finitely generated projective modules over its coordinate ring [1]. In topology, vector bundles over a compact Hausdorff space correspond to finitely generated projective modules over $C(X)$. In this talk, we will give an algebraic analogue for Swan's extension lemma which is needed to prove the Serre–Swan correspondence.

We call the following theorem Swan's Local–Global extension lemma.

Theorem 1. *Let X be normal. Let U be a neighborhood of x, and let s be a section of a vector bundle E over U. Then there is a section s' of E over X such that s' and s agree in some neighborhood of x (taken from [2]).*

The following is our main result.

Proposition 2. *Let A be a Noetherian normal domain with fraction field K, and let $X = \operatorname{Spec}(A)$. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of finite rank, and set $P = \Gamma(X, \mathcal{E})$. Fix a point $p \in X$, let $U \subseteq X$ be an open neighbourhood of p, and let $s \in \Gamma(U, \mathcal{E})$. Then there exist an effective Weil divisor*

$$D = \sum_{\operatorname{ht}(q)=1} n_q[q], \quad (n_q \in \mathbb{Z}_{\geq 0}),$$

such that $\operatorname{Supp}(D) \cap U = \varnothing$, and a global section $\widetilde{s} \in \Gamma\big(X, (\mathcal{E} \otimes \mathcal{O}_X(D))^{**}\big)$ whose restriction to U identifies with the original section s.

References

[1] Jean-Pierre Serre. Faisceaux algébriques cohérents. *Ann. of Math. (2)*, 61:197–278, 1955. doi:10.2307/1969915.

[2] Richard G. Swan. Vector bundles and projective modules. *Trans. Amer. Math. Soc.*, 105(2):264–277, 1962. doi:10.1090/S0002-9947-1962-0143225-6.