AGMA in the method of proving Minkowski's conjecture on Diophantine approximations, its applications, and generalizations

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We demonstrate methods from algebra, geometry, and mathematical analysis (AGMA) applied to prove results on Diophantine approximations, packings, and coverings related to Minkowski's conjecture on the critical determinant.

We investigate Minkowski balls, Minkowski spheres and Minkowski domains on the real plane. Minkowski spheres are boundaries of Minkowski balls. We consider balls of the form

$$D_p : \; |x|^p + |y|^p \leq 1, \; p \geq 1,$$

and call such balls *Minkowski balls*. Continuing this, we consider the following classes of Minkowski balls and circles (one-dimensional spheres).

- *Watson balls*: $|x|^p + |y|^p \leq 1$ for $p_0 > p \geq 2$;
- *Davis balls*: $|x|^p + |y|^p \leq 1$ for $p_0 > p \geq 2$;
- *Mordell–Chebyshev balls*: $|x|^p + |y|^p \leq 1$ for $p \geq p_0$.

Remark 1. This classification corrects the classification given in [2] and more accurately reflects the historical contributions of researchers.

Remark 2. Minkowski domains $2^m D_p$ are 2^m doubled Minkowski balls.

The algebraic part of the study concerns free modules, which are algebraic images of geometric lattices, and their invariants. This part (for brevity, we use lattice language), like geometry and analysis, is presented in theorems.

Theorem 3. *Let $m \in \mathbb{N}$ (see [3] for the case $m = 1$). The critical determinants of 2^m doubling balls D_p have a representation of the form*

$$\Delta_p^{(0)}(2^m D_p) = \Delta(p, \sigma_p)_{2^m D_p} = 2^{2m-1} \cdot \sigma_p, \; \sigma_p = (2^p - 1)^{1/p}, \tag{1}$$

$$\Delta_p^{(1)}(2^m D_p) = \Delta(p, 1)_{2^m D_p} = 4^{m-\frac{1}{p}} \frac{1 + \tau_p}{1 - \tau_p}, \; 2(1 - \tau_p)^p = 1 + \tau_p^p, \; 0 \leq \tau_p < 1. \tag{2}$$

And these are the determinants of the sublattices of index 2^m of the critical lattices of the corresponding balls D_p.

The extremal functions for packing of Minkowski balls and domains have the forms:

$$\Delta(2^m D_p) = \begin{cases} \Delta_p^{(1)}(2^m D_p), \; 1 < p \leq 2, \; p \geq p_0, \\ \Delta_p^{(0)}(2^m D_p), \; 2 \leq p \leq p_0; \end{cases} \tag{3}$$

References

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