On a total mixed curvature of closed curves

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Given a smooth closed curve γ with nowhere vanishing curvatures $k_1, \dots, k_j$ in $\mathbb{E}^n$, we consider the well-defined integral quantity

$$J_j(\gamma) = \oint_\gamma \sqrt{k_{j-1}^2 + k_j^2 + k_{j+1}^2}\, ds,$$

which arises as the total curvature or the total length of certain Frenet frame indicatrices of γ, viewed as smooth closed curves in Grassmann manifolds $G(j,n)$ embedded in $\mathbb{E}^{\binom{n}{j}}$; see [4].

For instance, for $\gamma \subset \mathbb{E}^3$ the quantity J_2, referred to as the total mixed curvature of γ, can be interpreted as the total curvature of the tangent indicatrix, the total curvature of the binormal indicatrix, or the length of the principal normal indicatrix associated with γ, where all three indicatrices are viewed as smooth closed curves in $S^2 \subset \mathbb{E}^3$; see [1, 2].

We address the problem of determining sharp lower bounds for $J_j(\gamma)$. It is known that if j is odd, then $J_j > 2\pi$, and this estimate is sharp [4]. On the other hand, if j is even, then the problem remains open and appears to be challenging even in the case of $j = 2$.

Problem 1. What is the infimum of J_2 over the space Ω of all smooth closed curves in $\mathbb{E}^n$, $n \geq 3$, with nowhere vanishing curvatures k_1 and k_2:

$$\inf_{\gamma \in \Omega} \oint_\gamma \sqrt{k_1^2 + k_2^2 + k_3^2}\, ds \ = ?$$

For curves with nonvanishing constant curvatures in $\mathbb{E}^4$, one has $J_2 \geq 2\sqrt{5}\,\pi$; see [6]. Furthermore, any circle can be realized as a limit curve for the space of smooth closed curves with nowhere vanishing curvatures k_1 and k_2 in $\mathbb{E}^4$, but the limit value of J_2 cannot be less than $2\sqrt{5}\,\pi$; see [5]. Thus, it was conjectured that the estimate $J_2 \geq 2\sqrt{5}\,\pi$ holds for all smooth closed curves with nowhere vanishing curvatures k_1 and k_2 in $\mathbb{E}^4$.

It turns out that the conjectured lower bound for J_2 must be weakened. Namely, the following statements hold true.

Proposition 2. *For an arbitrary smooth closed curve γ with nowhere vanishing curvatures k_1 and k_2 in $\mathbb{E}^n$, $n \geq 3$, there exists a smooth deformation which decreases the value of J_2. Thus, the desired infimum of J_2 can only be attained in the limit by curves that no longer belong to Ω.*

Proposition 3. *For any $\varepsilon > 0$, there exists a smooth closed curve γ with nowhere vanishing curvatures k_1 and k_2 in $\mathbb{E}^n$, $n \geq 3$, such that $J_2(\gamma) < 4\pi + \varepsilon$.*

Proposition 4. *For any $\varepsilon > 0$, there exists a smooth closed curve γ with nowhere vanishing curvatures k_1 and $k_2 \equiv \mathrm{const}$ in $\mathbb{E}^n$, $n \geq 3$, such that $J_2(\gamma) < 4\pi + \varepsilon$.*

The corresponding curves in Propositions 3–4 are constructed explicitly and independently.

We also note that Proposition 4, when compared with Proposition 3, illustrates the validity of the h-principle for closed curves with constant torsion in $\mathbb{E}^3$ established recently in [3].

References

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