

Telescoping transformations for reciprocal series of second-order recurrence sequences

Kunle Adegoke

Obafemi Awolowo University, Ile-Ife, Nigeria

E-mail: adegoke00@gmail.com

Robert Frontczak

Independent Researcher, Reutlingen, Germany

E-mail: robert.frontczak@web.de

Taras Goy

Vasyl Stefanyk Carpathian National University, Ivano-Frankivsk, Ukraine

E-mail: taras.goy@pnu.edu.ua

The aim of this paper is to develop a unified approach based on a discrete Abel-type summation formula (summation by parts). The main idea is to transform reciprocal series containing three or more factors in the denominator into expressions exhibiting a partial telescoping or alternating structure. This transformation reduces the complexity of the sums and makes it possible to derive families of identities in a uniform way. Our results may be viewed as a continuation and extension of the results obtained in [1], where several three-parameter families of finite reciprocal sums involving Horadam numbers were evaluated.

We consider sequences $w_n = w_n(a, b; p)$ defined by the recurrence relation

$$w_n = pw_{n-1} + w_{n-2}, \quad n \geq 2,$$

with initial values $w_0 = a$ and $w_1 = b$. This class includes, as special cases, the Fibonacci sequence $F_n = w_n(0, 1; 1)$ and the Lucas sequence $L_n = w_n(2, 1; 1)$, as well as several related sequences. We also use the associated sequences $u_n = w_n(0, 1; p)$ and $v_n = w_n(2, p; p)$.

Theorem 1. *Let r , s , and l be nonnegative integers. If m is odd, then*

$$\sum_{k=1}^{\infty} \frac{1}{w_{mk+s}w_{m(k+1)+s} \prod_{i=0}^{n-3} w_{r(k+i)+l}} = \frac{1}{u_m w_{m+s}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(w_{rk+l} + w_{r(k+n-2)+l})u_{mk}}{w_{m(k+1)+s} \prod_{i=0}^{n-2} w_{r(k+i)+l}}.$$

If m is even, then

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{w_{mk+s}w_{m(k+1)+s} \prod_{i=0}^{n-3} w_{r(k+i)+l}} = \frac{1}{u_m w_{m+s}} \sum_{k=1}^{\infty} (-1)^k \frac{(w_{rk+l} + w_{r(k+n-2)+l})u_{mk}}{w_{m(k+1)+s} \prod_{i=0}^{n-2} w_{r(k+i)+l}}.$$

By selecting appropriate values of r , m and l , we obtain from Theorem 1 the following classical-type and alternating identities involving Fibonacci and Lucas numbers:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{F_k F_{k+1} F_{k+2}} &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{F_{k+1}}, & \sum_{k=1}^{\infty} \frac{(-1)^k}{F_k F_{2k} F_{2k+2}} &= \sum_{k=1}^{\infty} (-1)^k \frac{F_{k+2} L_k}{F_{k+1}^2 L_{k+1}}, \\ \sum_{k=1}^{\infty} \frac{1}{F_k F_{k+1} F_{k+2} F_{k+3}} &= \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{F_{k+1} + 2F_k}{F_{k+1} F_{k+2} F_{k+3}}, \\ \sum_{k=1}^{\infty} \frac{(-1)^k}{F_k F_{k+1} F_{2k} F_{2k+2}} &= \sum_{k=1}^{\infty} (-1)^k \frac{F_{k+1} + 2F_k}{F_{k+1} F_{k+2} F_{2k+2}} L_k. \end{aligned}$$

REFERENCES

- [1] Kunle Adegoke, Robert Frontczak, and Taras Goy. Reciprocal series involving Horadam numbers. *Ukrainian Math. J.*, 75(3):335–346, 2023. doi:10.1007/s11253-023-02202-w.