On topologization of the extended bicyclic semigroup

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Every Hausdorff shift-continuous topology on the (extended) bicyclic semigroup is discrete [1, 3, 4]. Conditions under which a T_1-topology on the bicyclic monoid is discrete are given in [2].

Example 1. On the extended bicyclic semigroup $\mathcal{C}_\mathbb{Z}$ the following non-discrete T_1-topologies are constructed:

(i) a semigroup inverse topology;
(ii) a Baire left-continuous (right-continuous) topology;
(iii) a locally compact left-continuous (right-continuous) topology.

Proposition 2. *On the extended bicyclic semigroup $\mathcal{C}_\mathbb{Z}$ there does not exist a countably compact left-continuous (right-continuous) T_1-topology.*

We give conditions under which a T_1-topology on the extended bicyclic semigroup $\mathcal{C}_\mathbb{Z}$ is discrete.

Theorem 3. *A left-continuous (right-continuous) T_1-topology τ on $\mathcal{C}_\mathbb{Z}$ is discrete if and only if there exists a sequence $\{(i_n, j_n)\}_{n\in\mathbb{N}}$ of isolated points in $(\mathcal{C}_\mathbb{Z}, \tau)$ such that the sequence $\{i_n\}_{n\in\mathbb{N}}$ ($\{j_n\}_{n\in\mathbb{N}}$) is strictly increasing.*

Theorem 4. *A shift-continuous T_1-topology τ on $\mathcal{C}_\mathbb{Z}$ is discrete if and only if the space $(\mathcal{C}_\mathbb{Z}, \tau)$ contains an isolated point.*

A topological space X is called *quasi-regular* at $x \in X$ if for any open neighbourhood $U(x)$ of x in X there exists an open nonempty subset V of X such that $\mathrm{cl}_X(V) \subseteq U(x)$.

Theorem 5. *If the T_1-space of a topological inverse semigroup $(\mathcal{C}_\mathbb{Z}, \tau)$ has a point x such that $(\mathcal{C}_\mathbb{Z}, \tau)$ is quasi-regular at x, then τ is discrete.*

References

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