# Controllability problems for the parabolic equation in a half-plane controlled by the Dirichlet boundary condition with a bounded control

**Larissa Fardigola**
B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Nauky Ave., Kharkiv, 61103, Ukraine
*E-mail:* fardigola@ilt.kharkov.ua

**Kateryna Khalina**
B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Nauky Ave., Kharkiv, 61103, Ukraine
*E-mail:* khalina@ilt.kharkov.ua

Let $T > 0$, $\Omega = \big((0, +\infty) \times \mathbb{R}\big)$, $\Delta = (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$. Consider the following control system in a half-plane

$$w_t = \Delta w, \qquad (x_1, x_2) \in \Omega,\ t \in (0, T), \tag{1}$$

$$w\big(0, (\cdot)_{[2]}, t\big) = u((\cdot)_{[2]}, t), \qquad x_2 \in \mathbb{R},\ t \in (0, T), \tag{2}$$

$$w\big((\cdot)_{[1]}, (\cdot)_{[2]}, 0\big) = w^0, \qquad (x_1, x_2) \in \Omega, \tag{3}$$

where $u \in U[0, T]$ is a control,

$$U[0, T] = \left\{ \varphi \in L^\infty(\mathbb{R} \times (0, T)) \,\middle|\, \sup_{t \in [0,T]} |\varphi(\cdot, t)| \in L^2(\mathbb{R}) \right\}$$

is the set of admissible controls. The subscripts [1] and [2] associate with the variable numbers, e.g. $(\cdot)_{[1]}$ and $(\cdot)_{[2]}$ correspond to $x_1$ and $x_2$, respectively, if we consider $f(x)$, $x \in \mathbb{R}^2$. We investigate control system (1)–(3) in subspaces of Sobolev spaces $H^{-2s}(\mathbb{R}^2)$ with $s = 0, 1$. In particular, we consider $w(\cdot, t) \in L^2(\Omega)$, $t \in [0, T]$ and $w^0 \in L^2(\Omega)$. We study the approximate controllability problem for system (1)–(3).

**Definition 1.** A state $w^0 \in L^2(\Omega)$ is said to be *approximately controllable* to a target state $w^T \in L^2(\Omega)$ in a given time $T > 0$ if for each $\varepsilon > 0$, there exists $u_\varepsilon \in U[0, T]$ such that there exists a unique solution $w_\varepsilon$ to system (1)–(3) with $u = u_\varepsilon$ and

$$\big\| w_\varepsilon\big((\cdot)_{[1]}, (\cdot)_{[2]}, T\big) - w^T \big\|_{L^2} < \varepsilon.$$

Controllability problems for parabolic equations were studied both in bounded and unbounded domains. However, most of the papers studying these problems deal with domains bounded with respect to the spatial variables.

The case of a bounded domain is fully studied. It is well known that each initial state of control system (1)–(3) in a bounded domain $D \subset \mathbb{R}^n$ with the boundary $\partial D$ of class $C^2$ can be driven to zero for a given time $T > 0$. This result was obtained by using Carleman inequalities (see, e.g. [3]).

The case of an unbounded domain is essentially different. An initial state cannot be driven to an arbitrary target state, but only to a specific one that satisfies a certain relation with the initial state. For example, it has been proved in [4] that there is no initial data in any Sobolev space of negative order that may be driven to zero in finite time.

In most papers, controls belonging to $L^2$ were considered. But in our problem, we show that such controls are not appropriate to consider the approximate controllability property for $w^0 \in L^2(\Omega)$ and $w(\cdot, t) \in L^2(\Omega)$, $t \in [0, T]$. We consider the initial state $w^0 = 0 \in L^2(\Omega)$ and construct the control $u \in L^2(\mathbb{R} \times (0, T))$ with compact support by means of which $w^0 = 0$ is driven to the end state $w(\cdot, T)$ that does not belong to $L^2(\Omega)$. That is why we consider the narrower set of controls $U[0, T]$, i.e., we consider a specific subset of bounded controls in $L^2(\mathbb{R} \times (0, T))$. This condition guarantees that $w(\cdot, t) \in L^2(\Omega)$, $t \in [0, T]$, for any $w^0 \in L^2(\Omega)$.

We prove the following theorem.

**Theorem 2.** *Each state $w^0 \in L^2(\Omega)$ is approximately controllable to any target state $w^T \in L^2(\Omega)$ in a given time $T > 0$.*

In other words, each initial state $w^0 \in L^2(\Omega)$ of system (1)–(3) can be driven to an arbitrary neighbourhood of any target state $w^T \in L^2(\Omega)$ by choosing an appropriate control $u \in U[0,T]$.

The method of proving this theorem is constructive and it provides us with a numerical algorithm for constructing controls solving the approximate controllability problem for system (1)–(3) and for approximate end states which are reached by means of these controls. We consider the odd extension of $w$ and $w^0$ with respect to $x_1$ and obtain a new control system. Then we reduce the controllability problem for the 2-d parabolic equation controlled by the Dirichlet boundary condition to a finite family of controllability problems for the 1-d ones controlled by the same boundary conditions. For this we develop the state and the control in a new control system in the Fourier series with respect to a basis generated by Hermite functions. For solving the approximate controllability problem for the 1-d parabolic equation, we apply the method introduced in [1]. As a result we construct controls solving our problem and approximate end states which are reached by means of constructed controls. Finally, we analyse the solution to the control problem by applying the Fourier transform and its inverse.

The theorem and the numerical algorithm of solving the approximate controllability problem for system (1)–(3) are illustrated by an example.

All obtained results are published in [2].

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## References


[1] L. Fardigola and K. Khalina. Reachability and controllability problems for the heat equation on a half-axis. *J. Math. Phys. Anal. Geom.*, 15(1):57–78, 2019. doi:10.15407/mag15.01.057.

[2] L. Fardigola and K. Khalina. Approximate controllability problems for the heat equation in a half-plane controlled by the Dirichlet boundary condition with a bounded control. *J. Math. Phys. Anal. Geom.*, 22:34–65, 2026.

[3] A. V. Fursikov and O. Yu. Imanuvilov. *Controllability of Evolution Equations*, volume 34 of *Lecture Notes Series*. Research Institute of Mathematics, Global Analysis Research Center, Seoul National University, Seoul, 1996.

[4] S. Micu and E. Zuazua. On the lack of null-controllability of the heat equation on the half-space. *Port. Math. (N.S.)*, 58(1):1–24, 2001.