Emden–Fowler Type Equations in the Theory of Geodesic Mappings

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By a generalized Emden–Fowler equation one usually means an equation of the form

$$y'' + f(x)y^n = 0.$$

A considerable number of scientific works have been devoted to the study of this equation and related nonlinear differential models. In particular, significant contributions to the development of this field were made by researchers of Odesa I. I. Mechnikov National University, including V. M. Evtukhov, A. A. Stekhun, M. O. Bilozerova, and other scholars [2, 1].

The geometric properties of solutions to this equation have also been investigated in connection with applications in the theory of relativity. In particular, for $n = 2$, one obtains the Kustaanheimo–Qvist solution, which represents one of the solutions to the Einstein field equations:

$$R_{ij} - \frac{R}{2} g_{ij} = E_{ij}.$$

We investigate an Emden–Fowler type equation of the form

$$y'' + pyy' + qy^3 = 0.$$

Problems concerning geodesic mappings of a broad class of pseudo-Riemannian spaces can be reduced to this equation. These include, in particular, Einstein spaces, almost Einstein spaces, spaces whose degree of mobility with respect to geodesic mappings exceeds two, and many other types of special pseudo-Riemannian spaces [3, 4, 5].

A geodesic mapping is defined as a one-to-one correspondence between points of pseudo-Riemannian spaces under which geodesic lines of one space are mapped onto geodesic lines of another space. If such a mapping is not homothetic, it is called nontrivial.

The study of the properties of Emden–Fowler type equations has made it possible to establish that complete pseudo-Riemannian spaces for which this equation admits solutions do not allow nontrivial geodesic mappings.

References

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