

ON EXPLICIT REEB SPACES OF REAL ANALYTIC FUNCTIONS WHICH ARE NOT  
HOMEOMORPHIC TO ANY FINITE GRAPH

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Since the last century, *Reeb spaces* of (continuous real-valued) functions have been fundamental and important tools. They are in nice cases graphs (*Reeb graphs*: [10]). In theory of Morse functions and naturally generalized functions and applications to differential topology and geometric topology of manifolds, they have been also fundamental and important topological and combinatorial objects.

**Definition 1.** For a continuous real-valued function  $c : X \rightarrow \mathbb{R}$  on a topological space  $X$ , by the following equivalence relation  $\sim_c$  on  $X$ , we have the quotient space  $R_c := X/\sim_c$  and this is the *Reeb space* of  $c$ :  $x_1 \sim_c x_2$  if and only if they are in a same connected component of a same level set  $c^{-1}(y)$ . We have the quotient map  $q_c : X \rightarrow R_c$  and the unique continuous function  $\bar{c} : R_c \rightarrow \mathbb{R}$  with  $c = \bar{c} \circ q_c$ .

**Theorem 2** ([11]). *If  $X$  is a smooth closed and connected manifold and the critical value set of  $c$  is finite, then  $R_c$  has the structure of a finite and connected graph whose vertex set consists of all points  $v$  with  $q_c^{-1}(v)$  containing some critical points of  $c$  ([11, Theorem 3.1]).*

For more explicit cases, see [2] for example. We call a graph obtained in the rule above, the *Reeb graph* of a smooth function  $c : X \rightarrow \mathbb{R}$ , where  $X$  is with no boundary. The speaker has been interested in reconstruction of nice smooth or real algebraic functions whose Reeb graphs are isomorphic to given finite graphs and has contributed to this. We review related history and present our main problem (Problem 3).

- In [13], Sharko first considers reconstruction of a nice smooth function on a closed surface whose Reeb graph is isomorphic to a given finite graph of a certain natural class. Critical points of the function are isolated and around them, the function is represented by certain elementary polynomials or Morse. In [8], Masumoto and Saeki have extended this to arbitrary finite graphs and the critical sets of the functions may not be isolated.
- In [9], Michalak has considered reconstruction of Morse functions whose level sets with no critical point are disjoint unions of spheres and he has classified Reeb (di)graphs of Morse functions on each closed surface.
- The author has launched studies on reconstruction of nice smooth functions with given Reeb graphs, and as additional data, given level sets. Related studies on smooth functions on closed manifolds are [3, 6], for example. For the non-compact case, see [4] for example.
- The author has launched studies on reconstruction of explicit real algebraic functions with given Reeb graphs, in [5]. Here, a *real algebraic* map is the canonical embedding of a union of connected components of the zero set of a real polynomial map, or its composition with the canonical projection  $\pi_{k_1, k_2} : \mathbb{R}^{k_1} \rightarrow \mathbb{R}^{k_2}$  with  $k_1 > k_2 \geq 1$  and  $\pi_{k_1, k_2}(x_1, x_2) := x_1$ . It is a fundamental and important fact that in the real algebraic (analytic) category, we cannot use tools or methods such as bump functions, the partition of the unity, and so on. In smooth (differentiable) situations, we locally construct nice smooth functions and glue them together to have desired functions and we cannot apply this in algebraic (analytic) cases.

**Problem 3.** For (Reeb) spaces which are not finite graphs, can we reconstruct nice smooth functions?

Some examples of smooth functions which are not real analytic have been presented in [11, 12] for example. Different from such cases, we consider real analytic maps or smooth functions which are real analytic on dense subsets and each component of which is an elementary function.

**Theorem 4** ([7]). *There exists an infinite graph  $G_1$ , for any integer  $m \geq 2$ , there exists an  $m$ -dimensional smooth submanifold  $X_m \subset \mathbb{R}^{m+1}$  with no boundary which is represented as the zero set  $e_m^{-1}(0)$  of some real analytic and elementary function  $e_m : \mathbb{R}^{m+1} \rightarrow \mathbb{R}$ , and the following hold.*

- (1) *The Reeb space  $R_{\pi_{m+1,1}|_{X_m}}$  is regarded as the Reeb graph and isomorphic to  $G_1$ .*
- (2) *The quotient map  $q_{\pi_{m+1,1}|_{X_m}}$  is proper and the preimage of each point having no critical point of the function  $\pi_{m+1,1}|_{X_m}$  is diffeomorphic to  $S^{m-1}$ .*

According to [1], the Reeb space  $R_c$  of a smooth function  $c : X \rightarrow \mathbb{R}$  on a closed and connected manifold is a compact, connected and locally connected metrizable space (*Peano continuum*).

**Theorem 5** ([7]). *There exists a Peano continuum  $G_2$  which is not homeomorphic to any graph, for any pair  $(m_1, m_2)$  of integers  $m_1, m_2 \geq 2$ , there exists an  $(m_1 + m_2)$ -dimensional smooth compact submanifold  $X_{m_1, m_2} \subset \mathbb{R}^{m_1 + m_2 + 2}$  with no boundary represented as the zero set  $e_{m_1, m_2}^{-1}(0)$  of some smooth map  $e_{m_1, m_2} : \mathbb{R}^{m_1 + m_2 + 2} \rightarrow \mathbb{R}^2$  which is real analytic and each component of which is an elementary function outside some subset  $Z_{m_1, m_2} \subset \mathbb{R}^{m_1 + m_2 + 2}$  of Lebesgue measure 0, and the following hold.*

- (3) *For some point  $p_{G_2}$ ,  $G_2 - \{p_{G_2}\}$  is homeomorphic to a graph, the preimage  $q_{\pi_{m_1 + m_2 + 2, 1}|_{X_{m_1, m_2}}}^{-1}(p_{G_2})$  is  $\{0\} \subset \mathbb{R}^{m_1 + m_2 + 2}$ , and  $R_{\pi_{m_1 + m_2 + 2, 1}|_{X_{m_1, m_2} - \{0\}}}$  is regarded as the Reeb graph of  $\pi_{m_1 + m_2 + 2, 1}|_{X_{m_1, m_2} - \{0\}}$ .*
- (4) *The Reeb space  $R_{\pi_{m_1 + m_2 + 2, 1}|_{X_{m_1, m_2}}}$  is homeomorphic to  $G_2$ .*

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