

Weakly and strongly connected Markov graphs for maps on trees

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Let X be a finite combinatorial tree, and $\sigma: V(X) \rightarrow V(X)$ be a map from the vertex set of X to itself. The Markov graph $\Gamma(X, \sigma)$ is a directed graph having edges of X as its vertices with the existence of an arc $\{u, v\} \rightarrow \{x, y\}$ indicating the fact that $\{u, v\}$ “covers” $\{x, y\}$ under σ (meaning that x, y lie on a unique shortest path between $\sigma(u)$ and $\sigma(v)$ in X).

Recall that a digraph is called weakly connected provided the corresponding undirected graph is connected. The next results describe the structure of self-maps on trees with weakly connected Markov graphs (see [1]). We start with maps that produce weakly connected Markov graphs for some trees.

Proposition 1. *Let $n = |V| \geq 3$. Then for a map $\sigma: V \rightarrow V$, there exists a tree X on V such that $\Gamma(X, \sigma)$ is weakly connected if and only if σ is neither a constant map nor the identity map on V .*

A map $\sigma: V \rightarrow V$ is called nilpotent if for some $k \in \mathbb{N}$, the map σ^k is constant. A nilpotent map σ is path-nilpotent provided $|\sigma^{-1}(x) \setminus \{x\}| = 1$ for all $x \in \sigma(V)$. A permutation σ of V is called circular of length m if σ has exactly $|V| - m$ fixed points and it cyclically permutes its m non-fixed points. The next result characterizes maps that produce weakly connected Markov graphs for all trees.

Theorem 2. *Let $n = |V| \geq 2$. For a map $\sigma: V \rightarrow V$, the Markov graph $\Gamma(X, \sigma)$ is weakly connected for all trees X on V if and only if σ satisfies one of the following conditions:*

- (1) σ is a path-nilpotent map;
- (2) σ is a circular permutation of length $n - 1$;
- (3) σ is a permutation, and the periods of σ -periodic points with disjoint orbits are pairwise incomparable under division.

Recall that a digraph is called strongly connected provided it has a spanning closed walk. From Theorem 2 it follows that every cyclic permutation on a tree produces a weakly connected Markov graph. The next theorem describes trees which admit cyclic permutations with non-strongly connected Markov graphs (see [2]). A forest is called balanced if each of its trees has the same number of vertices.

Theorem 3. *For a tree X , there exists a cyclic permutation σ of $V(X)$ such that its Markov graph $\Gamma(X, \sigma)$ is not strongly connected if and only if there exists a proper spanning set of edges $E' \subset E(X)$ such that $X[E']$ is a balanced forest.*

Proposition 4. *Let $n = |V| \geq 3$. For a map $\sigma: V \rightarrow V$, the Markov graph $\Gamma(X, \sigma)$ is strongly connected for all trees X on V if and only if n is a prime number and σ is a cyclic permutation.*

Finally, we describe the structure of permutations that produce strongly connected Markov graphs for some trees.

Theorem 5. *Let $n = |V| \geq 5$. For a permutation $\sigma: V \rightarrow V$, there exists a tree X on V such that $\Gamma(X, \sigma)$ is strongly connected if and only if σ has at most $\frac{n-1}{2}$ fixed points.*

REFERENCES

- [1] Sergiy Kozerenko. Maps of combinatorial trees with weakly connected Markov graphs. *Util. Math.*, 123:35–52, 2025.
- [2] Sergiy Kozerenko. On strongly connected Markov graphs of maps on combinatorial trees. *Discrete Math. Lett.*, 15:31–38, 2025. doi:10.47443/dml.2024.218.