On F-planar mappings of quasi-Kähler spaces

Irina Kurbatova
ONU, Odesa, Ukraine
E-mail: irina.kurbatova27@gmail.com

Nadiia Konovenko
ONUT, Odesa, Ukraine
E-mail: ngkonovenko@gmail.com

A Riemannian space (V_n, g_{ij}, F_i^h) with metric tensor g_{ij} and affine structure F_i^h is called *quasi-Kähler* [1] if the conditions

$$F_\alpha^h F_i^\alpha = -\delta_i^h,$$
$$g_{i\alpha} F_j^\alpha = -g_{j\alpha} F_i^\alpha,$$
$$F_{i,j}^h = -F_{\alpha,\beta}^h F_i^\alpha F_j^\beta,$$

where the comma "," is the sign of the covariant derivative with respect to the connection of V_n. Quasi-Kähler spaces include well-known classes of almost complex manifolds, such as Kähler, K-, H-spaces.

It is easy to prove that a quasi-Kähler structure is integrable if and only if it is Kähler.

Consider the quasi-Kähler spaces (V_n, g_{ij}, F_i^h) and $(\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h)$. In the general coordinate system (x^i) the F-planar mapping [2]

$$(V_n, g_{ij}, F_i^h) \longrightarrow (\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h),$$

is characterized by the following basic equations:

$$\overline{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \psi_{(i}\delta_{j)}^h + \phi_{(i}F_{j)}^h,$$
$$F_i^h(x) = \overline{F}_i^h(x),$$
$$F_\alpha^h F_i^\alpha = -\delta_i^h,$$
$$F_{ij} = -F_{ji}, \qquad \overline{F}_{ij} = -\overline{F}_{ji},$$
$$F_{ij} = g_{i\alpha} F_j^\alpha, \qquad \overline{F}_{ij} = \overline{g}_{i\alpha} F_j^\alpha,$$
$$F_{i,j}^h = -F_{\alpha,\beta}^h F_i^\alpha F_j^\beta, \qquad F_{i|j}^h = -F_{\alpha|\beta}^h F_i^\alpha F_j^\beta,$$

where $\Gamma_{ij}^h, \overline{\Gamma}_{ij}^h$ are the Christoffel symbols of $V_n, \overline{V}_n$, respectively; $\psi_i(x)$, $\phi_i(x)$ are some covectors; brackets (i, j) denote the symmetrization operation; comma "," and vertical bar "|" are the signs of the covariant derivative with respect to the connection of V_n and $\overline{V}_n$, respectively.

An F-planar mapping is considered trivial if $\psi_i = \phi_i = 0$.

We have proved that for a nontrivial F-planar mapping, the vectors ψ_i and ϕ_i are related:

$$\psi_{\bar{i}} = -\phi_i, \qquad \phi_{\bar{i}} = \psi_i.$$

We agree to denote the affinor contraction operation as follows:

$$B_{\bar{i}\dots}^{\dots} = B_{\alpha\dots}^{\dots} F_i^\alpha, \quad B_{\dots}^{\bar{i}\dots} = B_{\dots}^{\alpha\dots} F_\alpha^i.$$

A number of geometric objects (both inhomogeneous and tensor-like) invariant under F-planar mappings are constructed.

An F-planar mapping of a quasi-Kähler space onto a flat Riemannian space is considered.

We have proved the following.

Theorem 1. *In order for a quasi-Kähler space (V_n, g_{ij}, F_i^h) to admit an F-planar mapping onto a flat space, it is necessary that the Riemann tensor in it satisfies the conditions*

$$R_{ijk}^h - R_{\bar{i}jk}^{\bar{h}} + R_{i\bar{j}\bar{k}}^h - R_{\bar{i}\bar{j}\bar{k}}^{\bar{h}} = \frac{4R}{n(n+2)}\Big[\delta_k^h g_{ij} - \delta_j^h g_{ik} - F_j^h F_{ik} + F_k^h F_{ij} + 2F_i^h F_{kj}\Big].$$

We will call quasi-Kähler spaces whose Riemann tensor satisfies the specified conditions *generalized F-flat.*

Theorem 2. *The class of generalized F-flat quasi-Kähler spaces (V_n, g_{ij}, F_i^h) is closed under F-planar mappings.*

References

[1] Alfred Gray and Luis M. Hervella. The sixteen classes of almost Hermitian manifolds and their linear invariants. *Annali di Matematica Pura ed Applicata*, 123(1):35–58, 1980. doi:10.1007/BF01796539.

[2] Josef Mikeš, Alena Vanžurová, Irena Hinterleitner, et al. *Differential Geometry of Special Mappings*. Palacký University Press, Olomouc, Czech Republic, 2nd edition, 2019. doi:10.5507/prf.19.24455358.

[3] Nikolai S. Sinyukov. *Geodesic Mappings of Riemannian Spaces*. Nauka, Moscow, 1979.