Invariance in time of α-convexity of order β

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The Hele–Shaw problem models the motion of a viscous fluid contained between two closely spaced parallel plates. Its evolution is described by a free boundary whose dynamics can be represented through conformal mappings satisfying the Polubarinova–Galin equation. This equation provides a powerful analytic framework linking geometric function theory to free-boundary dynamics in planar domains.

In this paper, we focus on the hereditary properties of α-convex functions of order β under the assumptions of the Hele–Shaw model. These functions form a two-parameter family that unifies and extends the classical starlike and convex classes. We analyse how the α-convexity of order β behaves under the conformal evolution dictated by the Hele–Shaw equation and identify sufficient conditions for which this property is preserved in time. The results establish a direct connection between analytic convexity and geometric stability, revealing how structural properties from univalent function theory control the invariance of evolving domains in Hele–Shaw-type flows.

In this paper we obtain a result which proves that the α-convexity of order β is preserved in time. The result is a generalisation of Theorem 2.3 from [1] to the case of α-convexity of order β. We recover Theorem 2.3 by taking $\beta = 0$.

Theorem 1. *Let $Q < 0$ and f_0 a univalent function on $\overline{U}$. Then, there exist $\alpha_0 \in (0,1)$ and $\beta \in [0,1)$ such that, if f is an α-convex function of order β on U, $\alpha < \alpha_0$, then the classical solution of the Polubarinova–Galin equation with the initial condition $f(z;0) = f_0(z)$ remains α-convex of order β during the existence time. Moreover, the existence time of the classical solution is infinite.*

References

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