

Differentiable structures on a line with $n + 1$ origins

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I extended the arguments from [2] to obtain a classification of differentiable structures on the line with $n + 1$ origins L_n up to a diffeomorphism which fixes Hausdorff-violating points.

Definition 1. Let τ be the standard topology on \mathbb{R} . Let L_n be a disjoint union of \mathbb{R} with the set $\{0_1, 0_2, \dots, 0_n\}$ endowed with the following topology:

$$\eta = \tau \cup \prod_{i=1}^n \{(W \setminus \{0\}) \cup \{0_i\} : 0 \in W \in \tau\}$$

with additional elements being the open neighborhoods of 0 in which 0 is replaced by 0_i .

The space L_n is locally Euclidean and T_1 , but not T_2 as the set $\text{Hv} := \{0, 0_1, 0_2, \dots, 0_n\}$ consists precisely of points for which the T_2 separation axiom fails to hold. Such points are often called *Hausdorff-violating*.

Definition 2. Let \mathcal{A} be a differentiable atlas on L_n . If \mathcal{A} consists of charts of the form $\{(U_i, \phi_i)\}_{i=0}^n$ where $U_i := \mathbb{R} \setminus \{0\} \cup \{0_i\}$ ($U_0 := \mathbb{R}$), $\phi_i(0_i) = 0$ ($\phi_0(0) = 0$) and ϕ_i maps $(0; +\infty) \rightarrow (0; +\infty)$, then \mathcal{A} is called the *minimal atlas* on L_n .

For any two charts in a minimal atlas the transition map is a monotone \mathcal{C}^r -diffeomorphism of $\mathbb{R} \setminus \{0\}$. Therefore there exists a unique *continuous extension* of $\phi_j \phi_i^{-1}$ to the homeomorphism of \mathbb{R} such that $0 \mapsto 0$. Let $T_{i,j}^{(\mathcal{A})}$ ($T_j^{(\mathcal{A})}$ if $i = 0$) denote such extension in some atlas \mathcal{A} .

In general, if f is a \mathcal{C}^r -diffeomorphism which acts on Hv by a permutation σ , then we have the following commutative diagrams and corresponding relations between extensions of transition maps.

$$\begin{array}{ccc}
 & U_j \xrightarrow{f} V_q & \\
 & \swarrow \phi_j \quad \searrow \psi_q & \\
 \mathbb{R} & \xrightarrow{f_{j,q}} & \mathbb{R} \\
 \uparrow T_{i,j}^{(\mathcal{A})} & \xrightarrow{\psi_j f \phi_q^{-1}} & \uparrow T_{q,p}^{(\mathcal{B})} \\
 \mathbb{R} & \xrightarrow{f_{i,p}} & \mathbb{R} \\
 & \swarrow \phi_i \quad \searrow \psi_p & \\
 & U_i \xrightarrow{f} V_p & \\
 & \swarrow \phi_i \quad \searrow \psi_p & \\
 \mathbb{R} & \xrightarrow{f_{i,p}} & \mathbb{R} \\
 & \xrightarrow{\psi_i f \phi_p^{-1}} & \\
 & \left\{ \begin{array}{l} f_{i,p} = T_{q,p}^{(\mathcal{B})} f_{j,q} T_{i,j}^{(\mathcal{A})} \\ T_{q,p}^{(\mathcal{B})} = f_{i,p} T_{j,i}^{(\mathcal{A})} f_{j,q}^{-1} \end{array} \right. & \\
 \end{array} \implies \begin{array}{ccc}
 \mathbb{R} & \xrightarrow{f_{0,\sigma(0)}} & \mathbb{R} \\
 \downarrow T_{0,i}^{(\mathcal{A})} & \xrightarrow{\psi_{\sigma(0)} f \phi_0^{-1}} & \downarrow T_{\sigma(0),\sigma(i)}^{(\mathcal{B})} \\
 \mathbb{R} & \xrightarrow{f_{i,\sigma(i)}} & \mathbb{R} \\
 & \xrightarrow{\psi_{\sigma(i)} f \phi_i^{-1}} & \\
 T_{\sigma(0),\sigma(i)}^{(\mathcal{B})} & = f_{i,\sigma(i)} T_{0,i}^{(\mathcal{A})} f_{0,\sigma(0)}^{-1} &
 \end{array}$$

For each $r \in \mathbb{N} \cup \{\infty\}$ let

- $\mathcal{W}_{\mathbb{R}} := \mathcal{W}^r(\mathbb{R}, 0)$ be the group of homeomorphisms of \mathbb{R} such that any element of $\mathcal{W}_{\mathbb{R}}$ maps 0 to itself and is a monotone \mathcal{C}^r -diffeomorphism of $\mathbb{R} \setminus \{0\}$;
- $\mathcal{D}_{\mathbb{R}} := \mathcal{D}^r(\mathbb{R}, 0)$ be the group of \mathcal{C}^r -diffeomorphisms of \mathbb{R} which fix 0. Note that $\mathcal{D}_{\mathbb{R}}$ is a subgroup of $\mathcal{W}_{\mathbb{R}}$.

Proposition 3. Let \mathfrak{A} be some differentiable structure on L_n . Then \mathfrak{A} contains a minimal atlas \mathcal{A} as a subatlas. Moreover, there is a one-to-one correspondence between minimal atlases and elements $\Xi^{(\cdot)} := (T_i)_{i=1}^n$ of the direct product $\prod_{i=1}^n \mathcal{W}_{\mathbb{R}}$.

If f fixes Hausdorff-violating points, the corresponding permutation σ is trivial and the theorem follows.

Theorem 4. Let \mathfrak{A} and \mathfrak{B} be two differentiable structures (maximal atlases) on L_n , then the following conditions are equivalent:

- (i) *There exists a diffeomorphism $(L_n, \mathfrak{A}) \rightarrow (L_n, \mathfrak{B})$ which fixes all Hausdorff-violating points;*
(ii) *For any pair of minimal atlases $\mathbf{A} \in \mathfrak{A}$ and $\mathbf{B} \in \mathfrak{B}$, elements $\Xi^{(\mathbf{A})}$ and $\Xi^{(\mathbf{B})}$ lie in the same orbit of the action*

$$\mathcal{D}_{\mathbb{R}} \times \prod_{i=1}^n \mathcal{D}_{\mathbb{R}} \times \prod_{i=1}^n \mathcal{W}_{\mathbb{R}} \rightarrow \prod_{i=1}^n \mathcal{W}_{\mathbb{R}}$$

$$(f_0, f_1, \dots, f_n) \cdot (T_1, \dots, T_n) \mapsto (f_1 T_1 f_0^{-1}, \dots, f_n T_n f_0^{-1})$$

The definitions of \mathcal{C}^r -structure, \mathcal{C}^r -atlas and \mathcal{C}^r -diffeomorphism can be found in the book by J. Lee [1] and the paper of F. Takens [3] with references therein.

REFERENCES

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