Lie-like structures over a product of vector spaces, with application to the Sheffer and Riordan groups

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The Riordan group was introduced in paper [6], and its study therein was significantly based upon the previous results from the umbral calculus, e.g. [5]. While the original motivation for defining the Riordan group came from enumerative combinatorics, this group also found applications in matrix theory, group theory, number theory, mathematical physics, computer science and other research areas in mathematical sciences. We refer to the monograph [7], solely devoted to the Riordan group, its extensions, deformations and applications.

The Riordan group consists of infinite lower triangular matrices $P = [p_{ni}]_{n,i=0,1,2,\dots}$ whose matrix elements p_{ni} satisfy the condition $\sum_{n=0}^{\infty} p_n(x)\xi^n = A(\xi)\big(1 - xB(\xi)\big)^{-1}$, where $p_n(x) = \sum_{i=0}^{n} p_{ni}\, x^i$, and $A(\xi)$ and $B(\xi)$ are formal power series, which depend on the matrix P. This definition provides alternative realisations of the Riordan group as one consisting of polynomial sequences $(p_n(x))_{n=0}^{\infty}$, or one consisting of pairs $(A(\xi), B(\xi))$ of formal power series. The product in the Riordan group is the usual product of infinite lower diagonal matrices. This product can be explicitly described as a certain group product of pairs of formal power series $(A(\xi), B(\xi))$. In particular, the Riordan group contains the subgroup of pairs $(1, B(\xi))$, in which the group product is the composition of formal power series. It should be noted that such a group of formal power series was introduced in the 1950s [3], and has been actively studied since then as an algebraic, geometric and topological object.

A close relative of the Riordan group is the Sheffer group (also called the exponential Riordan group). This group consists of infinite lower triangular matrices P for which the corresponding polynomial sequence $(p_n(x))_{n=0}^{\infty}$ (called a Sheffer sequence) has the exponential generating function $\sum_{n=0}^{\infty} \frac{1}{n!}\, p_n(x)\xi^n = A(\xi)\exp\big(xB(\xi)\big)$. The Sheffer polynomial sequences play an important role in probability, in particular, in the theory of Lévy processes, with applications found in mathematical finance.

The seminal paper [4] initiated rigorous studies of infinite-dimensional Lie groups and their Lie algebras. Such Lie groups and Lie algebras play an important role in mathematical physics. For example, the Lie group of local diffeomorphisms (with composition of diffeomorphisms as group operation) and its Lie algebra of vector fields are important for quantum physics. The study of infinite-dimensional Lie groups and Lie algebras remains an active research area.

In papers [1, 2], both the Riordan group and the Sheffer group have been studied as infinite-dimensional Lie groups and their Lie algebras have been identified, including the explicit form of the Lie bracket. In particular, it was shown in [2] that the Lie algebra of the Sheffer group consists of the following linear operators acting in polynomials: $\sum_{k=1}^{\infty} a_k D^k + X\sum_{k=2}^{\infty} b_k D^k$, where D is the operator of differentiation, X is the operator of multiplication by the variable of the polynomials, and a_k, b_k are constants. Since the operators X and D satisfy the commutation relation $DX - XD = 1$, they generate a Weyl algebra, which is fundamental for quantum field theory. In particular, the explicit form of the Lie bracket in the Lie algebra of the Sheffer group can be seen as a consequence of the commutation relation in the Weyl algebra [2].

The concepts of the Riordan group and the Sheffer group and the study of their Lie structures were extended to the multivariate case, and even to the case of infinitely many variables, cf. [7, Chapter 7] and [2]. Sheffer polynomials on an infinite-dimensional space play an important role in infinite-dimensional analysis, quantum probability and mathematical physics.

To apply Milnor's theory [4] to study Lie structures of the Sheffer and Riordan groups over an infinite-dimensional space in [2], some crucial assumptions were made about the topological structure of the underlying space.

The aim of this talk is to show that it is possible to develop a Lie-like theory of the Sheffer and Riordan groups over an arbitrary vector space, without any use of topology. This allows us not only to significantly simplify the arguments used in [2] but also extend them to a much wider class of underlying vector spaces, which are important for applications.

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