On non-existence of continuous decomposition of symmetric matrices into a linear combination of orthoprojectors

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Recall that a linear operator $P\colon \mathbb{R}^n \to \mathbb{R}^n$ is

- *symmetric* (selfadjoint) if $P^t = P$;
- *orthoprojector* if $P^t = P$ and $P^2 = P$.

For $k \geq 1$ denote by

- $\mathcal{P}^n_k$ the set of orthoprojectors onto subspaces of dimension k, i.e. having rank k;
- $\mathcal{P}^n = \cup_{i=0}^n \mathcal{P}^n_k$ the set of all orthoprojectors;
- Sym^n the set of all symmetric $n \times n$-matrices.

Clearly, $\mathcal{P}^n \subset Sym^n$. Moreover, $\mathcal{P}^n_k$ is naturally identified with the corresponding Grassmannian manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$.

There is an extensive literature discussing the problem of representing symmetric operators as linear combinations of orthoprojectors, i.e. for a given matrix $A \in Sym^n$ one looks for some numbers $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ and orthoprojectors $P_1, \dots, P_m \in \mathcal{P}^n$ such that

$$A = \lambda_1 P_1 + \cdots + \lambda_k P_k.$$

This problem has a lot of variations, say one can put a restriction on the number k of orthoprojectors and their ranges.

It can be reformulated as follows. Consider the map

$$\phi_k\colon \mathbb{R}^k \times (\mathcal{P}^n)^k \to Sym^n, \qquad \phi_k(\lambda_1, \dots, \lambda_k, P_1, \dots, P_k) = \lambda_1 P_1 + \cdots + \lambda_k P_k.$$

Then the question is to describe the image of this map.

Since every $A \in Sym^n$ has an orthonormal basis consisting of eigenvectors (spectral decomposition theorem), the map ϕ_k is surjective for $k \geq n$.

Note also that associating to a matrix A some $2n$-tuple $(\lambda_1, \dots, \lambda_k, P_1, \dots, P_k)$ such that $A = \phi_k(\lambda_1, \dots, \lambda_k, P_1, \dots, P_k)$ can be regarded as a *section* of ϕ_k, i.e. a map

$$s\colon Sym^n \to \mathbb{R}^k \times (\mathcal{P}^n)^k$$

such that $\phi_k \circ s = \mathrm{id}_{Sym^n}$. It is easy to see that ϕ_k admits a section if and only if it is surjective. However, such a section is not always *continuous*.

The aim of the talk is to discuss existence of *continuous* sections of ϕ_k and obstructions to constructing them. In particular we will prove the following

Theorem 1. *The map*

$$\phi_2\colon \mathbb{R}^2 \times \mathcal{P}^2_1 \times \mathcal{P}^2_1 \to Sym^2$$

is surjective, but it does not admit a continuous section.

In other words, it is not possible to associate to each symmetric 2×2 matrix $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ a pair of coefficients $\lambda_1(A), \lambda_2(A) \in \mathbb{R}$, and a pair of orthoprojectors $P_1(A), P_2(A) \in \mathcal{P}^2_1$ onto some lines in $\mathbb{R}^2$ so that

$$A = \lambda_1(A) P_1(A) + \lambda_2(A) P_2(A),$$

and these functions $\lambda_1, \lambda_2, P_1, P_2$ will be continuous in the coefficients of A.

The proof is based on the non-triviality of the second homotopy group $\pi_2(S^2) = \mathbb{Z}$ of the 2-sphere.

References