Admissible transformations and Lie symmetries of linear Schrödinger equations with complex-valued time-independent potentials

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The study of Lie symmetries for Schrödinger equations was started in the 1970s with the linear case, and it has been continued up to now for more complicated classes of linear Schrödinger equations; see [2, 3, 5, 4] and references therein. Admissible transformations and Lie symmetries of $(1+n)$-dimensional linear Schrödinger equations with (in general) time-dependent complex-valued potentials, which are of the general form

$$i\psi_t + \psi_{aa} + V(t,x)\psi = 0, \tag{1}$$

were studied in [2] for $n = 1$ and in [3] for arbitrary $n \in \mathbb{N}$. Here t and $x = (x_1, \dots, x_n)$ are the real independent variables, ψ is the complex dependent variable and V is an arbitrary smooth complex-valued potential depending on t and x. Here and in what follows the index a runs from 1 to n, and we assume summation over repeated indices. In [2], the notion of uniformly semi-normalized classes of differential equations was introduced and the algebraic method of group classification for such classes was suggested. The results were developed in [3]. The equivalence groupoid $\mathcal{G}^\sim_\mathcal{F}$ of the class $\mathcal{F}$ of the above equations was computed, and it was thus shown that this class is uniformly semi-normalized with respect to the linear superposition of solutions. Hence, the group classification of $\mathcal{F}$ reduces to the classification of specific low-dimensional subalgebras of the associated equivalence algebra. The group classification problem for the class $\mathcal{F}$ was completely solved in [2, 3] for space dimensions one and two, respectively.

In the present work, we consider the subclass $\mathcal{F}'$ of the class $\mathcal{F}$ that consists of the equations of the form (1) with time-independent complex-valued potentials. Based on the description of $\mathcal{G}^\sim_\mathcal{F}$ in [3], we construct the equivalence group $G^\sim_{\mathcal{F}'}$ of this subclass and describe its equivalence groupoid via classifying the admissible transformations within this subclass. This gives the first example in the literature on applying the method of furcate splitting to classifying admissible transformations within a non-normalized class of differential equations.

Theorem 1. *The equivalence group $G^\sim_{\mathcal{F}'}$ of the class $\mathcal{F}'$ consists of the point transformations in the space with the coordinates $(t, x, \psi, \psi^*, V, V^*)$ whose (t, x, V)-components are of the form*

$$\tilde t = \lambda_1 t + \lambda_0, \quad \tilde x = \varepsilon|\lambda_1|^{1/2}x + \nu, \quad \tilde\psi = \mathrm{e}^{i\lambda_3 t + i\lambda_2 + \lambda_5 t + \lambda_4}\hat\psi, \quad \tilde V = \frac{\hat V}{|\lambda_1|} + \frac{\lambda_3 - i\lambda_5}{\lambda_1},$$

where λ_0, …, λ_5 and ν are real constants with $\lambda_1 \neq 0$, and $\varepsilon := \pm 1$.

Theorem 2. *A generating (up to the $G^\sim_{\mathcal{F}'}$-equivalence and the linear superposition of solutions) set of admissible transformations $(V, \Phi, \tilde V)$ for $\mathcal{F}'$ is the union of the families*

$$\mathcal{T}_{1F} := \big(x_n^{-2}F([x_1:\ldots:x_n]) - x_ax_a,\ \Phi_1,\ \tilde x_n^{-2}F([\tilde x_1:\ldots:\tilde x_n])\big),$$
$$\Phi_1\colon\ \ \tilde t = \frac12\tan 2t,\ \ \tilde x_a = \frac{x_a}{\cos 2t},\ \ \tilde\psi = |\cos 2t|^{n/2}\mathrm{e}^{i\tan(2t)|\boldsymbol{x}|^2/2}\psi,$$

$$\mathcal{T}_{2F} := \big(x_n^{-2}F([x_1:\ldots:x_n]) + x_ax_a,\ \Phi_2,\ \tilde x_n^{-2}F([\tilde x_1:\ldots:\tilde x_n])\big),$$
$$\Phi_2\colon\ \ \tilde t = \frac14\mathrm{e}^{4t},\ \ \tilde x_a = \mathrm{e}^{2t}x_a,\ \ \tilde\psi = \mathrm{e}^{i|\boldsymbol{x}|^2/2 - nt}\psi,$$

$$\mathcal{T}_{3\alpha U} := \big(U(x_2,\dots,x_n) + i\alpha x_1 + x_1,\ \Phi_{3\alpha},\ U(\tilde x_2,\dots,\tilde x_n) + i\alpha\tilde x_1\big),$$
$$\Phi_{3\alpha}\colon\ \ \tilde t = t,\ \ \tilde x_1 = x_1 - t^2,\ \ \tilde x_a = x_a,\ a\neq 1,\ \ \tilde\psi = \mathrm{e}^{-itx_1 + (i+\alpha)t^3/3}\psi,$$

$$\mathcal{T}_{4\alpha U\kappa} := \big(U(x_2,\dots,x_n) + i\alpha x_1 - x_1^2,\ \Phi_{4\alpha\kappa},\ U(\tilde x_2,\dots,\tilde x_n) + i\alpha\tilde x_1 - \tilde x_1^2\big),$$

$$\Phi_{4\alpha\kappa}\colon\ \ \tilde t=t,\ \ \tilde x_1=x_1+2\kappa\cos 2t,\ \ \tilde x_a=x_a,\ a\neq1,\ \ \tilde\psi=\mathrm e^{\kappa\sin t\,(-2ix_1-2i\kappa\cos 2t-\alpha)}\psi,$$

$$\mathcal T_{5\alpha U\kappa\nu}:=\big(U(x_2,\dots,x_n)+i\alpha x_1+x_1^2,\ \Phi_{5\alpha\kappa\nu},\ U(\tilde x_2,\dots,\tilde x_n)+i\alpha\tilde x_1+\tilde x_1^2\big),$$

$$\Phi_{5\alpha\kappa\nu}\colon\ \ \tilde t=t,\ \ \tilde x_1=x_1+2\kappa\mathrm e^{2t}+2\nu\mathrm e^{-2t},\ \ \tilde x_a=x_a,\ {\scriptstyle a\neq1},\ \ \tilde\psi=\mathrm e^{(\kappa\mathrm e^{2t}-\nu\mathrm e^{-2t})(2i(x_1+\kappa\mathrm e^{2t}+\nu\mathrm e^{-2t})-\alpha)}\psi,$$

where $[x_k:\dots:x_n]$ *with* $k\in\{1,\dots,n\}$ *denotes homogeneous coordinates in the projective space of dimension* $n-k$*,* F *is a general sufficiently smooth complex-valued function in this space with* $k=1$*,* U *is a general sufficiently smooth complex-valued function of* $(x_2,\dots,x_n)$*,* $\alpha,\kappa,\nu\in\mathbb R$*,* $\kappa\neq0$ *in the fourth family, and* $(\kappa,\nu)\neq(0,0)$ *in the fifth family. Moreover, in the last two families,* $\alpha\neq0$ *or*

$$x_2U_2+\dots+x_nU_n+2U\neq4\epsilon(x_2^2+\dots+x_n^2),$$

where $\epsilon=-1$ *and* $\epsilon=1$ *for the fourth and the fifth families, respectively.*

Since the class $\mathcal F'$ possesses no normalization properties, we use results of [2, 3] and the above theorems to exhaustively solve the group classification problems for this subclass in space dimensions one and two up to the general point equivalence and up to the equivalences generated by the corresponding equivalence groups. But the group classification problem for $\mathcal F'$ and, even more so, for $\mathcal F$ in space dimension three are much more cumbersome than their counterparts for $n=1$ and $n=2$. Since the group classification problem for the class $\mathcal F$ has not been solved for $n>2$ and, we think, will not be completely solved in the near future, we directly solve the group classification problem for the class $\mathcal F'$ with $n=3$ up to both $G^\sim_{\mathcal F'}$- and $\mathcal G^\sim_{\mathcal F'}$-equivalences using an *original algebraic version of the method of furcate splitting*. Then *we reduce the classification of essential Lie symmetry extensions in the class* $\mathcal F'$ *with* $n=3$ *to the classification of nonzero subalgebras of the extended Euclidean algebra* $\bar{\mathrm e}(3)$. Based on the well-known complete list of nonzero inequivalent (with respect to inner automorphisms) subalgebras of $\bar{\mathrm e}(3)$ presented in [1], we use an optimized procedure to construct a complete list of $G^\sim_{\mathcal F'}$-inequivalent essential Lie-symmetry extensions within the class $\mathcal F'$.

References

[1] W. I. Fushchych, L. F. Barannyk, and A. F. Barannyk. *Subgroup Analysis of Galilei and Poincaré Groups and Reduction of Nonlinear Equations*. Naukova Dumka, Kyiv, 1991.

[2] Célestin Kurujyibwami, Peter Basarab-Horwath, and Roman O. Popovych. Algebraic method for group classification of (1 + 1)-dimensional linear Schrödinger equations. *Acta Appl. Math.*, 157:171–203, 2018. arXiv:1607.04118. doi:10.1007/s10440-018-0169-y.

[3] Célestin Kurujyibwami, Dmytro R. Popovych, and Roman O. Popovych. Algebraic method of group classification for semi-normalized classes of differential equations. 2024. arXiv:2408.16897, 42 pp. doi:10.48550/arXiv.2408.16897.

[4] A. G. Nikitin. Symmetries of Schrödinger equation with scalar and vector potentials. *J. Phys. A: Math. Theor.*, 53(45):455202, 2020. arXiv:2005.10305. doi:10.1088/1751-8121/abb956.

[5] A. G. Nikitin and T. M. Zasadko. Group classification of Schrödinger equations with position dependent mass. *J. Phys. A: Math. Theor.*, 49(36):365204, 2016. arXiv:1603.00890. doi:10.1088/1751-8113/49/36/365204.