

# Homotopy structure of orbits of smooth functions lifted to an orientable double cover

Bohdan Mazhar

Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

*E-mail:* mazhar@imath.kiev.ua

Sergiy Maksymenko

Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

*E-mail:* maks@imath.kiev.ua

Let  $M$  be a surface (smooth compact 2-manifold) and  $f: M \rightarrow \mathbb{R}$  be a smooth function. By smooth we mean  $\mathcal{C}^\infty$ . We study the homotopy types of orbits  $\mathcal{O}(f)$  with respect to the natural right action of the smooth diffeomorphisms group on the space of smooth functions  $\mathcal{C}^\infty(M, \mathbb{R}) \times \mathcal{D}(M) \rightarrow \mathcal{C}^\infty(M, \mathbb{R})$ , given by  $(f, h) \mapsto f \circ h$ .

Consider the subclass  $\mathcal{F}(M, \mathbb{R}) \subset \mathcal{C}^\infty(M, \mathbb{R})$  of functions that, in particular, contains Morse functions, which are dense in  $\mathcal{C}^\infty(M, \mathbb{R})$ . It is well-known (see [1]) that, for each  $f \in \mathcal{F}(M, \mathbb{R})$ , the homotopy types of  $\mathcal{O}(f)$  are weak, i.e., they are completely described by the collection of the homotopy groups  $\pi_k \mathcal{O}(f)$ ,  $k \geq 1$ . Only a few cases yield non-trivial higher homotopy groups  $\pi_k \mathcal{O}(f)$ ,  $k \geq 2$ , while the structure of the fundamental group  $\pi_1 \mathcal{O}(f)$  is highly non-trivial and is of the main interest.

Let  $\tilde{N}$  be an orientable surface with a smooth orientation-reversing involution  $\tau: \tilde{N} \rightarrow \tilde{N}$  without fixed points, and  $N = \tilde{N}/\tau$  be the quotient, a non-orientable surface. Then there is a smooth double covering map  $p: \tilde{N} \rightarrow N$ , and the lifted function  $\tilde{f} = f \circ p$ , which lies in the class  $\mathcal{F}(\tilde{N}, \mathbb{R})$ . One can uniquely map each  $h \in \mathcal{D}(N)$  to the orientation-preserving lift  $\tilde{h} \in \mathcal{D}(\tilde{N})$  such that  $p \circ \tilde{h} = h \circ p$ . This lifting induces a well-defined mapping of orbits  $\theta: \mathcal{O}(f) \rightarrow \mathcal{O}(\tilde{f})$ .

**Theorem 1.** *Map  $\theta$  is  $\pi_1$ -injective, i.e., induces a group monomorphism  $\pi_1 \mathcal{O}(f) \rightarrow \pi_1 \mathcal{O}(\tilde{f})$ .*

**Theorem 2.** *Assume Euler characteristic  $\chi(N) < 0$ , and function  $f \in \mathcal{F}(N, \mathbb{R})$ .*

(1) *There exists (see e.g. [2]) a finite collection  $\{S_\alpha\}$  of subsurfaces of  $N$  whose elements are diffeomorphic either to disks  $D_i$ , cylinders  $C_j$ , or Möbius bands  $M_k$ , such that*

$$\pi_1 \mathcal{O}(f) \cong \prod_{\alpha} \pi_1 \mathcal{O}(f|_{S_\alpha}) \cong \prod_i \pi_1 \mathcal{O}(f|_{D_i}) \times \prod_j \pi_1 \mathcal{O}(f|_{C_j}) \times \prod_k \pi_1 \mathcal{O}(f|_{M_k}).$$

(2) *There exists a finite collection  $\{\tilde{S}_\beta\}$  of subsurfaces of  $\tilde{N}$  whose elements are diffeomorphic either to disks  $\tilde{D}_u$ , or cylinders  $\tilde{C}_v$ . Moreover, there is a subcollection of cylinders  $\{\tilde{C}_k\} \subset \{\tilde{C}_v\}$  which are in one-to-one correspondence with the collection of Möbius bands  $\{M_k\}$ , such that*

$$\begin{aligned} \pi_1 \mathcal{O}(\tilde{f}) &\cong \prod_{\beta} \pi_1 \mathcal{O}(f|_{\tilde{S}_\beta}) \cong \prod_u \pi_1 \mathcal{O}(\tilde{f}|_{\tilde{D}_u}) \times \prod_v \pi_1 \mathcal{O}(\tilde{f}|_{\tilde{C}_v}) \\ &\cong \prod_i (\pi_1 \mathcal{O}(f|_{D_i}))^2 \times \prod_j (\pi_1 \mathcal{O}(f|_{C_j}))^2 \times \prod_k \pi_1 \mathcal{O}(\tilde{f}|_{\tilde{C}_k}). \end{aligned}$$

*In particular,  $\#\{\tilde{D}_u\} = 2\#\{D_i\}$  and  $\#\{\tilde{C}_v\} = 2\#\{C_j\} + \#\{M_k\}$ .*

## REFERENCES

- [1] Sergiy Maksymenko. Deformations of functions on surfaces by isotopic to the identity diffeomorphisms. *Topology Appl.*, 282:107312, 2020. doi:10.1016/j.topol.2020.107312.
- [2] Bohdan Mazhar and Sergiy Maksymenko. Smooth functions that split a Klein bottle into two Möbius bands. 2025. doi:10.48550/arXiv.2508.19636.