Invariant idempotent $*$-measures generated by generalized iterated function systems

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Iterated Function Systems (IFSs), introduced by John Hutchinson [1], provide a powerful tool for constructing self-similar fractal objects, including sets, probability measures, and their generalizations. The concept of IFS was later extended to generalized iterated function systems (GIFSs). In this setting, for a metric space X, maps of the form $X^m \to X$ are considered. Such systems were introduced by Alexandru Mihail and Radu Miculescu [3]. They defined the notion of an invariant element in the hyperspace $\exp(X)$ associated with these systems and proved its existence and uniqueness under natural contractivity conditions.

A GIFS on X of order m is a finite set of maps $g_i\colon X^m \to X$, $i = 1, \dots, n$, where g_i is a *generalized Matkowski contraction of degree* m. The last means that there exists nondecreasing $\varphi\colon [0, \infty) \to [0, \infty)$ such that $\varphi^{(k)}(t) \to 0$ as $k \to \infty$, and

$$d(g_i(x), g_i(y)) \le \varphi(d(x, y)), \quad x, y \in X^m,$$

where X^m is endowed with the maximum metric. For results concerning invariant objects appearing from GIFSs, see [4].

In [5], GIFSs acting on the space of probability measures were studied. In particular, it was shown that, under suitable contractivity conditions, given GIFS $\{g_1, g_2, \dots, g_n\}$ of order m and probabilities $p_i \ge 0$ with $\sum_{i=1}^{n} p_i = 1$, there exists an invariant Hutchinson measure, that is, a measure $\mu \in P(X)$ satisfying

$$\mu = \sum_{i=1}^{n} p_i \cdot P(g_i)(\mu \otimes \mu \otimes \cdots \otimes \mu).$$

We extend the results of [2], where the existence of invariant idempotent $*$-measures was established, to the case of GIFSs. In particular, for a given GIFS and a triangular norm $*$, we prove the existence and uniqueness of an invariant $*$-measure. Furthermore, we modify the notion of GIFS by considering maps not only of the form $X^m \to X$, but also maps defined on the G-symmetric power of a space, that is, $SP^m_G(X) \to X$.

Theorem 1. *Let (X, d) be a compact metric space and let $g_i\colon SP^m_G(X) \to X$, $i = 1, \dots, n$, be SP^m_G-generalized Matkowski contractions, where G is a subgroup of the symmetric group S_m. Let $*$ be a triangular norm and $\alpha = \vee_{i=1}^{n} \alpha_i * \delta_i$ be an idempotent $*$-measure on $\{1, 2, \dots, n\}$. Then there exists a unique invariant $*$-measure for the given SP^m_G-GIFS and α.*

We also note that our proof of the existence and uniqueness of invariant idempotent $*$-measures does not rely on metrization of the space of measures and is relatively simple.

References

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