Geometry vs. Topology: Classifying Curves via Zariski Pairs

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The interplay between geometry and topology is one of the most fascinating aspects of algebraic geometry. While two plane algebraic curves can look identical from a local or combinatorial perspective, their global embeddings into the projective plane can behave in entirely different ways.

The Zariski pairs serve as the perfect manifestation of this phenomenon, showing exactly where the pure algebraic data diverge from the topological reality.

The theory originated from this classical observation of Oscar Zariski—that two algebraic curves may share the same combinatorics and local singularity structure, while having non-homeomorphic embeddings in the projective plane.

Since the 1990s, the subject has developed into a rich interaction between algebraic geometry, topology, singularity theory, braid groups, arithmetic geometry, and computational methods, serving as a powerful framework for the classification of algebraic curves. A major turning point occurred with the work of Enrique Artal Bartolo in "Sur les couples de Zariski" [4]. In this paper, Bartolo systematically constructed Zariski pairs (using Alexander polynomials) and showed that combinatorics alone does not determine the topology of the complement of a plane curve. This work established a modern framework for studying the embedded topology of algebraic curves through topological invariants. This direction was expanded in Zariski pairs, fundamental groups and Alexander polynomials [7], where the topology of complements and fundamental groups became central tools. The paper develops techniques for distinguishing sextic curves with identical singularities and strongly influenced later work on line arrangements.

Ichiro Shimada introduced lattice-theoretic methods in "A note on Zariski pairs" [16]. This work connects Zariski pairs with K3 surfaces and brings arithmetic and algebraic-geometric methods into the subject.

The year 2000 brought further deep arithmetic-geometric perspectives into the field. An example of such progress can be found in [13], where arithmetic properties of elliptic K3 surfaces were related directly to embedded topology.

A new type of sextic Zariski pair was presented by Artal Bartolo, Carmona, Cogolludo, and Tokunaga in "Sextics with singular points in special position" [11], showcasing curves with triple points lying in special positions.

In the early 2000s, braid monodromy became one of the strongest invariants in the field. In "Braid monodromy and topology of plane curves" [8], braid monodromy was shown to encode refined topological information capable of distinguishing curves with identical combinatorics. This approach was further developed in "Effective invariants of braid monodromy" [9], where computable braid monodromy invariants were introduced. At the same time, work on arrangements explicitly demonstrated that combinatorics does not determine topology.

In "Topology and combinatorics of real line arrangements" [10], the authors constructed arrangements with identical incidence structures but different embeddings in the projective plane. This result became fundamental for the study of Zariski pairs of line arrangements.

During the 2010s, several new invariants appeared in "A linking invariant for algebraic curves" [15], "An arithmetic Zariski pair of line arrangements with non-isomorphic fundamental group" [12], and "Torsion divisors of plane curves with maximal flexes and Zariski pairs" [6]. Recent research shifted toward realization spaces, conic-line arrangements, and computational methods. In Zariski pairs of conic-line arrangements of degrees 7 and 8 via fundamental groups [3], new Zariski pairs were constructed using van Kampen techniques and Coxeter groups. Later, in "The realization space of a certain conic-line arrangement of degree 7 and a π_1-equivalent Zariski pair" [1], the authors showed that even arrangements with identical fundamental groups may still form Zariski pairs. This demonstrated that π_1 itself is not always sufficient for classification. There is also an investigation of Tokunaga's arrangements. One of them is shown in the following figure.

The modern viewpoint is summarized also in "Topology of complex plane curves: braid monodromy, local and global problems" [5], where braid monodromy is presented as a unifying framework connecting singularity theory, topology of complements, arithmetic geometry, realization spaces, and classification problems.

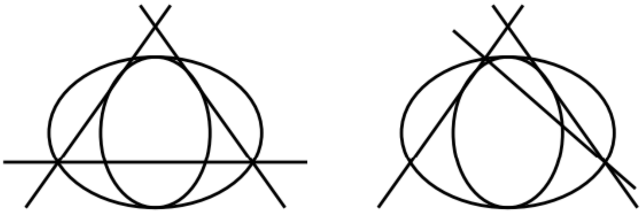
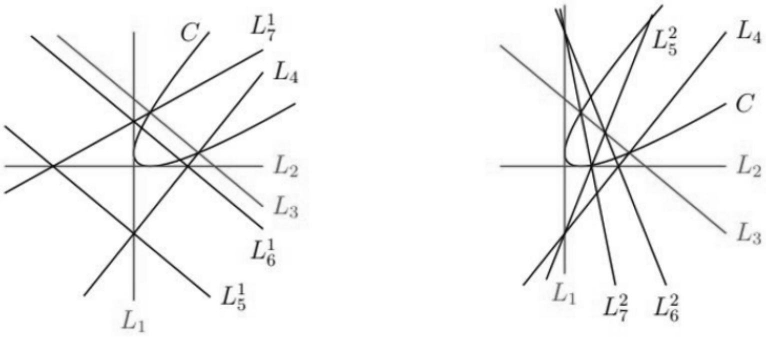


FIGURE 0.1. Tokunaga's arrangement

In my talk, I will merge all the above findings and background into a deep discussion that also includes fundamental group computations and software construction and utilization. Our recent work is summarized in "Detecting Zariski pairs by algorithms and computational classification in conic line arrangements" [2], where algorithmic and combinatorial techniques are used to classify arrangements and detect candidate Zariski pairs systematically. Motivated by known examples of Zariski pairs featuring one conic and 7 lines (in [14]), our findings establish a systematic inductive algorithm to classify $(n, 1)$-arrangements. In the talk, I will discuss how this approach proves that no Zariski pairs exist for $n \leq 4$. We will also present our newly constructed $(2, 3)$-arrangements, demonstrating that even curves with isomorphic fundamental groups can form Zariski pairs. I will talk on further plans to optimize these computational methods to handle a larger number of lines and to extend our combinatorial equivalence framework to support multiple conics. An example of a Zariski pair from [14] is depicted in the following figure (one conic and 7 lines).



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