Inversor of digits in the B-representation of numbers on the unit interval and its automodel and integro-differential properties

Olha Bondarenko
Ukrainian State University named after Mykhailo Drahomanov
E-mail: `o.i.bondarenko@udu.edu.ua`

Nataliia Vasylenko
Ukrainian State University named after Mykhailo Drahomanov
E-mail: `n.m.vasylenko@udu.edu.ua`

Let $A = Z = \{0, \pm 1, \pm 2, \dots\}$ be an alphabet (a set of digits), and let $L = A \times A \times \dots$ be the set (space) of sequences of elements of the alphabet. Let (Θ_n) be a two-sided sequence of positive real numbers ($n \in Z$) such that

$$0 < \sum_{n=1}^{\infty} \Theta_{-n} \equiv u < 1, \quad 0 < \sum_{n=0}^{+\infty} \Theta_n \equiv v < 1, \quad u + v = 1;$$

and let (b_n) be another two-sided sequence defined by the sequence (Θ_n), namely

$$b_n \equiv \sum_{i=-\infty}^{n-1} \Theta_i = b_{n-1} + \Theta_{n-1}.$$

Theorem 1. *For any number $x \in (0;1)$ there exists a unique finite set of integers*

$$(\alpha_1, \alpha_2, \dots, \alpha_m)$$

or a unique sequence $(\alpha_n) \in L$ such that one of the following equalities holds:

$$x = b_{\alpha_1} + \sum_{k=2}^{m} b_{\alpha_k} \prod_{i=1}^{k-1} \Theta_{\alpha_i} \equiv \Delta^B_{\alpha_1 \alpha_2 \dots \alpha_m (\emptyset)}, \tag{1}$$

$$x = b_{\alpha_1} + \sum_{k=2}^{\infty} b_{\alpha_k} \prod_{i=1}^{k-1} \Theta_{\alpha_i} \equiv \Delta^B_{\alpha_1 \alpha_2 \dots \alpha_k \dots}. \tag{2}$$

Symbolic representation of a number x by equality (1) or (2) is called the *B-representation* of this number, and $\alpha_n = \alpha_n(x)$ is its nth digit. Due to the uniqueness of the B-representation, the value $\alpha_n = \alpha_n(x)$ is a correctly defined function of the number x. Let us note that the B-representation of numbers has zero redundancy (each number possesses a unique B-representation).

Definition 2. The *inversor of digits* in the B-representation of numbers is defined as the function I given by the equalities

$$I(\Delta^B_{\alpha_1 \alpha_2 \dots \alpha_n \dots}) = \Delta^B_{[-\alpha_1][-\alpha_2] \dots [-\alpha_n] \dots}, \quad I(\Delta^B_{\alpha_1 \alpha_2 \dots \alpha_n (\emptyset)}) = \Delta^B_{[-\alpha_1][-\alpha_2] \dots [-\alpha_n] (\emptyset)}.$$

Lemma 3. *The inversor I of digits in the B-representation of numbers has the following properties:*

1) $I(\Delta^B_{(0)}) = \dfrac{b_0}{1 - \Theta_0} = \dfrac{u}{1 - \Theta_0}$;

2) I is a continuous and strictly decreasing function.

Theorem 4. *The graph Γ_I of the inversor I is an N-self-affine set, that is,*

$$\Gamma_I = \bigcup_{i \in \mathbb{Z}} f_i(\Gamma_I),$$

where the affine mappings f_i have the form

$$f_i : \begin{cases} x' = \Theta_i x + b_i, \\ y' = \Theta_{-i} y + b_{-i}. \end{cases}$$

The self-affine dimension of the graph of the inversor is the solution of the equation

$$\sum_{n=-\infty}^{+\infty} \begin{vmatrix} \Theta_n & 0 \\ 0 & \Theta_{-n} \end{vmatrix}^{\frac{x}{2}} = 1. \tag{3}$$

In particular, when $\Theta_n = \Theta_{-n}$, then equation (3) takes the form

$$\sum_{n=-\infty}^{\infty} \Theta_n^x = 1.$$

When $\sum_{n=0}^{\infty} \Theta_n^x = 1$, then equation (3) takes the form

$$\frac{1}{2^x} + \sum_{n=1}^{\infty} \frac{1}{2^{nx}} = 1 \quad \Leftrightarrow \quad \frac{1}{2^x} + \frac{1}{2^x - 1} = 1.$$

When $\Theta_n = \Theta_{-n} = \dfrac{1}{2^{n+2}}$ for $n \geq 1$ and $\Theta_0 = \dfrac{1}{2}$, then equation (3) takes the form

$$\Theta_0^x + 2\sum_{n=1}^{\infty} \Theta_n^x = 1.$$

Theorem 5. *If there is such n that $\Theta_n \neq \Theta_{-n}$, then the inversor is a singular function (a function which is continuous, non-constant, and has a derivative equal to zero almost everywhere in the sense of Lebesgue measure), and when $\Theta_n = \Theta_{-n}$ for all $n \in Z$ the equation $I(x) = -x$ holds.*

Theorem 6. *For the inversor $I(x)$ on the interval $[0, 1]$ the following equality holds:*

$$\int_0^1 I(x)\,dx = \frac{\sum\limits_{i\in\mathbb{Z}} b_i\Theta_i}{1 - \sum\limits_{i\in\mathbb{Z}} \Theta_i^2}.$$

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