On Generalized Graham–Kohr Extension Operators, Loewner Chains, and Janowski Subordination in the Unit Ball

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We study generalized Graham–Kohr type extension operators associated with normalized locally univalent functions on the unit disk $U \subset \mathbb{C}$ and their extensions to holomorphic mappings on the unit ball $B^n \subset \mathbb{C}^n$. These operators play an important role in extending geometric properties from one complex variable to several complex variables and include, as a particular case, the classical Roper–Suffridge operator [7].

Let f be a locally univalent function defined on the unit disk $\mathbb{U}$, and let $\gamma_n \in [0,1]$ and $\omega_n \in \left[0, \frac{1}{2}\right]$. We consider the family of extension operators

$$\Theta_{n,\gamma_n,\omega_n}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\gamma_n} \left(f'(z_1)\right)^{\omega_n} z' \right),$$

where $z = (z_1, z') \in \mathbb{C}^n$, $z' = (z_2, \ldots, z_n)$, and $n \geq 2$. When $\omega_n = \frac{1}{2}$ and $\gamma_n = 0$, the above operator reduces to the classical Roper–Suffridge extension operator. We show that $\Theta_{n,\gamma_n,\omega_n}$ maps the class of spirallike functions of type β (denoted by $\hat{S}_\beta$) into the class $S^0(\mathbb{B}^n)$ of mappings having parametric representation on $\mathbb{B}^n$ [3]. Furthermore, we prove that if f is a normalized univalent Bloch function on $\mathbb{U}$, then $\Theta_{n,\gamma_n,\omega_n}(f)$ is a Bloch mapping on the unit ball $\mathbb{B}^n$ [9].

We further investigate a more general extension operator introduced by M. Aron [1]. For parameters $\alpha \in [0,1]$, $\beta \in [0, \frac{1}{2}]$, and $\gamma \geq 1$, we consider

$$\Psi^{\gamma}_{n,\alpha,\beta}(f)(z) = \left(f(z_1), \tilde{z} \left[\frac{f(z_1)}{z_1}\right]^{\alpha} \left[1 + \frac{1}{\gamma}\left(\frac{z_1 f'(z_1)}{f(z_1)} - 1\right)\right]^{\beta} \right),$$

where $z = (z_1, \tilde{z}) \in \mathbb{C}^n$, $\tilde{z} = (z_2, \ldots, z_n)$, and $n \geq 2$. We investigate preservation properties of this operator for several geometric subclasses of univalent functions; see [5, 4, 6] for related background. In particular, for real numbers a, b satisfying $|1-a| < b \leq a$, we establish sufficient conditions on the parameters α, β, γ ensuring that

$$\Psi^{\gamma}_{n,\alpha,\beta}(S^*(a,b)) \subset S^*(a,b,B^n), \qquad \Psi^{\gamma}_{n,\alpha,\beta}(AS^*(a,b)) \subset AS^*(a,b,B^n).$$

In addition, for suitable choices of the parameters, we establish subordination results for subclasses of starlike functions [2, 8].

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