Geometry of encoding of numbers by means of a redundant alphabet in problems of function theory with fractal properties

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We consider representations of numbers from the interval $[0, \frac{r}{s-1}]$ with a natural base $s \geq 2$ and alphabet $A_r \equiv \{0, 1, \dots, r\}$, where $s \leq r \leq 2s-2$, namely

$$x = \Delta^{r_s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \equiv \frac{\alpha_1}{s} + \frac{\alpha_2}{s^2} + \ldots + \frac{\alpha_n}{s^n} + \ldots$$

The case $r = 2s-2$ requires special attention, since for $r > 2s-2$ all numbers possess a continuum of distinct representations, whereas for $s \leq r < 2s-2$ there exists a continuum set of numbers having unique representations. In the boundary case $r = 2s-2$ there is a unique subcase $s = 2 = r$, in which the points of a countable set have countably many representations, while all remaining points have continuum many representations.

Lemma 1. *If $s = 2 = r$, then the cylinders*

$$\Delta^{r_s}_{c_1\ldots c_{m-1}i} = [c, \frac{1}{s^m} + c],$$

$$\Delta^{r_s}_{c_1c_2\ldots c_{m-1}[i+1]} = [c + \frac{1}{s^m}, \frac{2}{s^m} + c],$$

$c = \sum_{i=1}^{m-1} \frac{c_i}{2^i}$, *have the specific intersection property*

$$\Delta^{r_s}_{c_1\ldots c_{m-1}i} \cap \Delta^{r_s}_{c_1\ldots c_{m-1}[i+1]} = \Delta^{r_s}_{c_1\ldots c_m i2} = \Delta^{r_s}_{c_1\ldots c_{m-1}[i+1]0}.$$

The specific overlap structure of neighboring cylinders described in Lemma 1 is another distinctive feature of this case.

Henceforth, let $s = 2 = r$.

Theorem 2. *The function f, defined by the equation $f(x = \Delta^3_{\alpha_1\ldots\alpha_n\ldots}) = \Delta^{r_s}_{\alpha_1\ldots\alpha_n\ldots}$, is continuous at points having a unique representation in the classical ternary numeral system and discontinuous at ternary-rational points: $\Delta^3_{c_1\ldots c_{m-1}c_m(0)} = \Delta^3_{c_1\ldots c_{m-1}[c_m-1](2)}$. Moreover, it possesses fractal level sets.*

Theorem 3. *The range of the complex-valued function $\varphi(t) = \sum_{n=1}^{\infty} 2^{-n}\varepsilon_{\alpha_n(t)}$ of a real argument $t = \Delta^3_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$, where ε_0, ε_1, ε_2 are the cubic roots of unity, is a spiderweb-type curve having self-similar fractal dimension $\log_2 3$, which coincides with its Hausdorff–Besicovitch dimension.*

Theorem 4. *Let (ξ_n) be a sequence of independent random variables taking values 0, 1, and 2 with probabilities p_0, p_1, and p_2 respectively, where $\max\{p_0, p_1, p_2\} \neq 1$. Then the random variable $\xi = \Delta^{r_s}_{\xi_1\xi_2\ldots\xi_n\ldots}$ has: 1) an absolutely continuous distribution if and only if $p_1 = \frac{1}{2}$ or $p_0 = p_2 = \frac{1}{2}$; 2) a purely singular distribution in all remaining cases.*

We note that an explicit expression for the distribution function of the random variable ξ is still unknown. In this case, the problem is of a combinatorial nature.

Theorem 5. *Let (τ_n) be a sequence of random variables taking values 0, 1, and 2, forming a Markov chain with nonzero initial probabilities p_0, p_1, p_2, and transition probability matrix $\|p_{ik}\|$. Then the random variable $\tau = \Delta^{r_s}_{\tau_1\tau_2\ldots\tau_n\ldots}$ may have purely discrete, purely singular, or purely absolutely continuous distributions, as well as distributions that are mixtures of discrete and continuous components* [1].

References

[1] Mykola V. Pratsiovytyi and Sofiia P. Ratushniak. Singular distributions of random variables with independent digits of representation in numeral system with natural base and redundant alphabet. *Matematychni Studii*, 63(2):199–209, 2025. doi:10.30970/ms.63.2.199-209.