Stratified Hamiltonian Flows on Immersions of the Projective Plane

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Let M be the image of an immersion $g\colon \mathbb{RP}^2 \to \mathbb{R}^3$, considered as a stratified space whose strata are determined by the singular set of the immersion.

A *simple stratified Hamiltonian flow (SSH-flow)* on M is a flow satisfying the following conditions:

- the flow has no singular points on the one-dimensional strata;
- for each two-dimensional stratum, the flow is the image of the Hamiltonian flow on a region of S^2 and the corresponding Hamiltonian function is a simple Morse function;
- in a neighborhood of every zero-dimensional stratum, the restrictions of the vector field to the adjacent one-dimensional strata are locally generated by Morse functions.

There exist two non-homeomorphic surfaces that are immersions of the projective plane with one triple point and a set of double points with three components — Boy's surface and Girl's surface [1]. The topological structure of flows with gradient dynamics on these surfaces was described in [2, 3]. The study of the structure of simple SH-flows is similar to the study of pro-Hamiltonian flows on non-orientable surfaces [5, 4].

Theorem 1. *On Boy's and Girl's surfaces, there exist 5 distinct structures of simple stratified Hamiltonian flows without internal saddle points.*

The qualitative structure of such flows can be described combinatorially by means of oriented Reeb graphs enriched with additional data associated with the singular strata.

The *distinguishing graph* of a simple SH-flow is defined as its oriented Reeb graph together with the cyclic ordering of the edges adjacent to the vertices corresponding to singular strata of the immersion.

Two distinguishing graphs are said to be equivalent if there exists a graph isomorphism preserving:

- the orientation of all edges;
- the types of vertices;
- the cyclic order of the edges incident to the distinguished vertices.

Theorem 2 (Criterion of Topological Equivalence)**.** *Simple stratified Hamiltonian flows are trajectory topologically equivalent if and only if their distinguishing graphs are equivalent.*

References

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