Structure of gradient flows with a degenerate node on the sphere with holes

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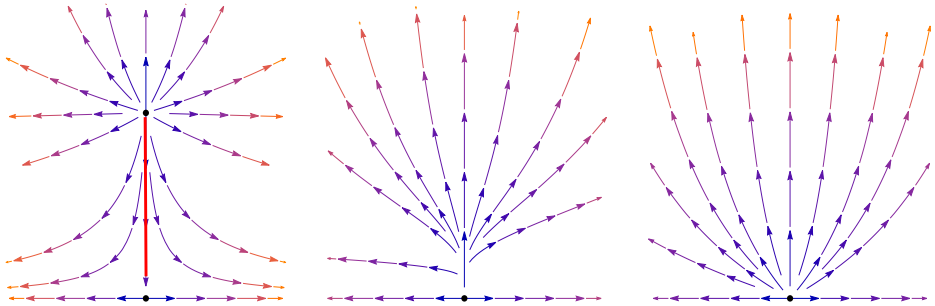
There are two types of degenerated nodes on the boundary that arise at $a = 0$ for such families of vector fields:

1) $V_1(x, y, a) = \{x, y^2 + ay\}$, $y \geq 0$, if the node is a source (HN_+ bifurcation, see Fig. 0.1);
2) $V_2(x, y, a) = \{-x, -y^2 - ay\}$, $y \geq 0$, if the node is a sink (HN_- bifurcation).

Here x, y are local coordinates, a is a real parameter.

In Fig. 0.1 a) the vector field before the bifurcation is shown ($a = -1$), in Fig. 0.1 b) – the vector field at the moment of bifurcation ($a = 0$), and in Fig. 0.1 c) – the vector field after the bifurcation ($a = 1$).



(a) (b) (c)

Figure 0.1. HN_+ bifurcation

Similar bifurcation can happen only on the boundary of a compact surface. In particular, both before and after the bifurcation the vector fields are Morse vector fields. A flow generated by the vector field in the moment of bifurcation is called an *HN-flow*.

We research construction of complete topological invariants for flows generated by vector fields before the moment of bifurcation — Morse flows, as a matter of fact. It is a *distinguishing graph* which is a generalization of a *chord diagram* represented in our previous work [1].

The algorithm of construction is made as follows. We take a Morse flow on a sphere with a finite number of holes, highlight all the stable manifolds of saddle points of the flow and mark all the boundary trajectories as well. Then, we cut out small enough neighbourhoods of all the sources in this flow. We obtain a specific discrete topological graph in which we have just to colour singular points (vertices of the graph) and trajectories (edges of the graph):

1) its edges representing boundaries of neighbourhoods of sources have a second colour (they are dashed), its edges representing stable manifolds of saddles in the interior of the sphere have a first (thin black) colour, the other edges have a third (thick black) colour;
2) the sinks of the flow lying on the boundary of the sphere have a white colour and the other vertices have a black colour.

Theorem 1. *HN-flows are topologically equivalent if and only if their distinguishing graphs are isomorphic; moreover, that isomorphism preserves orientations of the trajectories and colours of all the vertices and the edges.* [2]

Also the code for such flows was constructed. It is a specific finite sequence of symbols which corresponds to a certain distinguishing graph and thus represents a certain HN-flow with regard to a homeomorphism.

Finally, all the HN-flows on the sphere with one, two and three holes were counted provided that the number of all singular points of the flow is at most 6. So, for instance, there are 6 HN_+-flows with 5 singular points and 24 HN_+-flows with 6 singular points on D^2.

References

[1] Alexandr Prishlyak, Illia Ovtsynov. Optimal codimension one gradient flows on closed surfaces. *Proceedings of the International Geometry Center*, 18 (1) : 69–101, 2025.

[2] Alexandr Prishlyak, Illia Ovtsynov. The structure of Morse and codimension one gradient flows on the sphere with holes. *Memoirs on Differential Equations and Mathematical Physics*, 97 : 137–158, 2026.