On solutions of the matrix equation $AX+YB=C$ over an elementary divisor domain

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Let R be an elementary divisor domain with identity $e \neq 0$. We introduce the following notations: $\mathrm{R}_{m,n}$ is the set of $m \times n$ matrices over R, I_n is the identity $n \times n$ matrix, $0_{m,n}$ is the zero $m \times n$ matrix, and D^* is the classical adjoint matrix for a nonsingular matrix $D \in \mathrm{R}_{n,n}$, i.e., $D^*D = DD^* = (\det D)I_n$.

In an elementary divisor domain R all finitely generated ideals are principal. So, for two given elements $a, b \in \mathrm{R}$ there exists a greatest common divisor $d = (a, b)$ and elements $x, y \in \mathrm{R}$ satisfying the Bezout identity $ax + by = d$ (see [4]). Thus, every matrix A over an elementary divisor domain R admits a diagonal reduction, that is, if $A \in \mathrm{R}_{m,n}$ and $\operatorname{rank} A = r$, then there exist invertible matrices $P \in GL(m, \mathrm{R})$ and $Q \in GL(n, \mathrm{R})$ such that $U_A A V_A = S_A = \operatorname{diag}(a_1, a_2, \ldots, a_r, 0, \ldots, 0)$ is a diagonal matrix, where $a_1, a_2, \ldots, a_r$ are all nonzero and $a_i \mid a_{i+1}$ (divides) for all $i = 1, 2, \ldots, r-1$. The matrix S_A is called the Smith normal form of the matrix A.

Consider the matrix equation

$$AX + YB = C, \tag{1}$$

where $A \in \mathrm{R}_{m,m}$, $B \in \mathrm{R}_{n,n}$ and $C \in \mathrm{R}_{m,m}$ are given matrices and let $X, Y \in \mathrm{R}_{m,n}$ be unknown matrices. Solving matrix equation (1) over fields and commutative rings is one of the most important research problems, which has applications in many areas of mathematics and its applications.

Roth [10] proved that equation (1) is solvable over a field $\mathrm{R} = \mathbb{F}$ if and only if matrices $M_C = \begin{bmatrix} A & C \\ 0_{n,m} & B \end{bmatrix}$ and $M_0 = \begin{bmatrix} A & 0_{m,n} \\ 0_{n,m} & B \end{bmatrix}$ are equivalent. It is easy to verify that equation (1) is compatible over a field if and only if $\operatorname{rank} M_0 = \operatorname{rank} A + \operatorname{rank} B$.

Jones [6] proved that Roth's compatibility condition for equation (1) is true for the ring of analytic functions over the field of complex numbers $\mathbb{C}$. Gustafson in [5] proved that equation (1) over a commutative ring K with unity is compatible if and only if matrices M_C and M_0 are equivalent over K. If $\mathrm{K} = \mathbb{F}[\lambda]$ is the ring of polynomials over a field $\mathbb{F}$, then in works [1, 2, 3, 7, 8, 9, 11] conditions were investigated under which the matrix equation $A(\lambda)X(\lambda) - Y(\lambda)B(\lambda) = C(\lambda)$ over $\mathbb{F}[\lambda]$ admits "minimal" solutions over $\mathbb{F}[\lambda]$, i.e., solutions such that $\deg X(\lambda) < \deg B(\lambda)$ (or $\deg Y(\lambda) < \deg A(\lambda)$).

We note that the review of the literature and comments on current research is given in the extended paper [12]. In this report we investigate the solvability of the Sylvester-type matrix equation $AX + YB = C$ over an elementary divisor domain.

Theorem 1. *Let $A \in \mathrm{R}_{m,m}$ and $B \in \mathrm{R}_{n,n}$ be nonsingular matrices and $(\det A, \det B) = d$. Then for arbitrary matrix $C \in \mathrm{R}_{m,n}$ the matrix equation*

$$AX + YB = dC \tag{2}$$

is solvable over R. *Further, let $u, v \in \mathrm{R}$ such that $u(\det A) + v(\det B) = d$. Then for arbitrary $t \in \mathrm{R}$ and for arbitrary matrix $P \in \mathrm{R}_{m,n}$ the pair of matrices*

$$X = A^*C(u + t(\det B)) + A^*P(\det B) \quad \text{and} \quad Y = (v - t(\det A))CB^* - (\det A)PB^*$$

is the general solution of equation (2).

Corollary 2. *Let $A \in \mathrm{R}_{m,m}$ and $B \in \mathrm{R}_{n,n}$ be nonsingular matrices and $C \in \mathrm{R}_{m,n}$. Further, let $(\det A, \det B) = d$. If $C = dC_1$, then the matrix equation $AX + YB = C$ is solvable over* R.

Further, let $u, v \in \mathrm{R}$ such that $u(\det A) + v(\det B) = d$. Then for arbitrary $t \in \mathrm{R}$ and for arbitrary matrix $P \in \mathrm{R}_{m,n}$ the pair of matrices

$$X = A^*C_1(u + t(\det B)) + A^*P(\det B) \quad \text{and} \quad Y = (v - t(\det A))C_1B^* - (\det A)PB^*$$

is the general solution of equation $AX + YB = C$.

Corollary 3. *Let $A \in \mathrm{R}_{m,m}$ and $B \in \mathrm{R}_{n,n}$ be nonsingular matrices and $(\det A, \det B) = e$. Then for arbitrary matrix $C \in \mathrm{R}_{m,n}$ the matrix equation $AX + YB = C$ is solvable over* R.

Further, let $u, v \in \mathrm{R}$ *such that* $u(\det A) + v(\det B) = e$. *Then for arbitrary* $t \in \mathrm{R}$ *and for arbitrary matrix* $P \in \mathrm{R}_{m,n}$ *the pair of matrices*

$$X = A^*C(u + t(\det B)) + A^*P(\det B) \quad \text{and} \quad Y = (v - t(\det A))CB^* - (\det A)PB^*$$

is the general solution of equation $AX + YB = C$.

Proposition 4. *Let* $A \in \mathrm{R}_{m,m}$ *and* $B \in \mathrm{R}_{n,n}$ *be nonsingular matrices and* $C \in \mathrm{R}_{m,n}$. *Further, let* $(\det A, \det B) = d$. *If the matrix equation* $AX+YB = C$ *is solvable over* R, *then* $A^*CB^* \equiv 0 \pmod{d}$.

Theorem 5. *Let* $A \in \mathrm{R}_{m,m}$ *and* $B \in \mathrm{R}_{n,n}$ *be nonsingular matrices and* $C \in \mathrm{R}_{m,n}$. *Further, let* $S_A = \mathrm{diag}(e, \ldots, e, a)$ *and* $S_B = \mathrm{diag}(e, \ldots, e, b)$ *be the Smith normal forms of matrices* A *and* B *respectively and* $(\det A, \det B) = d$. *If* $A^*CB^* \equiv 0 \pmod{d}$, *then the matrix equation* $AX + YB = C$ *is solvable over* R.

Problem 6. When are the conditions of Proposition 4 sufficient for the existence of solutions of the equation $AX + YB = C$ over an elementary divisor domain R?

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