On the Grothendieck embedding problem of linear closed subspaces into Banach spaces revisited

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The problems concerned with estimation of the dimension of linear closed functional subspaces in $L_p(M, d\mu)$, $p > 1$ $(\neq 2)$ (in particular, in $L_p(0, 1; d\lambda)$), are of long-time interest in analysis, being related to their many applications in operator and approximation theories, in dynamical systems theory and other applied fields. As an example, one can recall here a central problem in Banach space theory: to classify the complemented subspaces of $L_p(M, d\mu)$ up to isomorphism; the finite-dimensional analogue is to find, for any given $S_p \subset L_p(M, d\mu)$, a description of the finite-dimensional spaces which are S_p-isomorphic to S_p-complemented subspaces of $L_p(M, d\mu)$. These problems were thoroughly studied before, in particular the finite-dimensional versions of the complemented subspaces problem for $L_p(M, d\mu)$, yet in neither case is their classification far from being closed.

It was also observed that it sometimes happens that the finite-dimensional version of an infinite-dimensional problem leads to a theory [3, 4], giving rise to new interesting aspects of the infinite-dimensional theory. Recall here the problem of describing the subspaces of $L_p(M, d\mu)$ which embed isomorphically into a "smaller" space, namely the space l_p, for which a fairly good answer is found. Assuming a space M is endowed with a probability measure μ, absolutely continuous with respect to a probabilistic measure ν, that is $d\mu = h\, d\nu$ for a strictly positive measurable function $h\colon M \to \mathbb{R}_+$ for which $\int_M h\, d\nu = 1$, we can induce, for fixed $1 < p < \infty$, an isometry $J_h^{(p)}$ from $L_p(M, d\mu)$ onto $L_q(M, d\nu)$ for any $q > p > 1$. In particular, the next result gives useful information about chosen *a priori* finite-dimensional full-support subspaces $S_p \subset L_p(M, d\mu)$.

Theorem 1. *Let μ be a probability measure on M and let S_p be an N-dimensional subspace of $L_p(M, d\mu)$, $1 < p < \infty$, with full support. Then there is a density $h > 0$ so that the image $J_h^{(p)} S_p$ has a basis $\{\varphi_1, \varphi_2, \dots, \varphi_N\} \subset L_2(M, h\, d\mu)$, which is orthonormal in $L_2(M, h\, d\mu)$ and such that $\sum_{j=1}^N |\varphi_j|^2 = N$.*

Moreover, just recently an interesting result [2] was announced concerning the Grothendieck-type embedding theorem for Bergman spaces. Namely, let $A_p(\Omega) = \mathrm{Hol}(\Omega) \cap L_p(\Omega)$ be a classical Bergman space of holomorphic functions on a bounded domain $\Omega \subset \mathbb{C}^n$, p-integrable on Ω with respect to the Lebesgue measure. Then the following theorem holds.

Theorem 2. *For any bounded domain $\Omega \subset \mathbb{C}^n$, if $1 \le p < q \le \infty$, E is a closed subspace of $A_p(\Omega)$ and $E \subset A_q(\Omega)$, then $\dim E < \infty$.*

Concerning a general problem of estimating the dimension of a linear closed topological subspace $S_p^{(q)} \subset L_p(M, d\mu) \hookrightarrow L_q(M, d\nu)$, identically embedded into the Banach space $L_q(M, d\nu)$, $q > p+\alpha > 1$ $(\neq 2)$, $\alpha \in (0, 1)$, we stated the following result.

Theorem 3. *Let a linear closed topological subspace $S_p^{(q)} \subset L_p(M, d\mu)$, $p+\alpha > 1$ $(\neq 2)$, be identically embedded into a Banach space $L_q(M, d\nu)$ for $q = 2+(p+\alpha-2)2^m > p+\alpha > 1$ $(\neq 2)$, $\alpha \in (0, 1)$, where $d\mu = h\, d\nu$ is a probability measure, absolutely continuous with respect to a probability measure $d\nu$ on M and satisfying the conditions $\int_M h\, d\nu = 1$ and $\int_M h^{(p+\alpha)/\alpha}\, d\nu < \infty$ for some $\alpha \in (0, 1)$. Then the dimension $\dim S_p^{(q)} = N \in \mathbb{N}$ of the closed subspace $S_p^{(q)} \subset L_p(M, d\mu)$ proves to satisfy the inequality*

$$N \left(\frac{\Gamma\left(\frac{N}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{\sqrt{\pi}\, \Gamma\left(\frac{N+q}{2}\right)} \right)^{2/q} \le K^2_{p,q(\alpha,m)}$$

for some fixed constant $K_{p,q(\alpha,m)} > 0$. *If the degree* $q \to \infty$, *the dimension* $\dim S_p^{(\infty)} \leq K^2_{p,q(\alpha,\infty)}$, *being a priori finite.*

Remark 4. Taking into account the estimation of the dimension $\dim S_p^{(q)} = N \in \mathbb{N}$ of a linear closed subspace $S_p^{(q)} \subset L_p(M, d\mu) \hookrightarrow L_q(M, d\nu)$ for $q = 2 + (p + \alpha - 2)2^m > p + \alpha > 1$ ($\neq 2$), $\alpha \in (0, 1)$, obtained in Theorem 3, it is interesting to analyse its interpretation and possible relationship to the known result of G. Pisier [3]:

Theorem 5. *In the space* $L_2(0, 1)$ *there exists a linear infinite-dimensional closed subspace* $G \subset L_2(0, 1)$, *which is a closed subspace of every* $L_p(0, 1)$, $1 \leq p < \infty$, *and for which the corresponding norms are proportional, that is, for arbitrary* $1 \leq p < \infty$ *there exist constants* $\gamma_p > 0$ *such that for any* $g \in G$ *the norms* $\|g\|_p = \gamma_p \|g\|_2$.

In the case when $\alpha \to 0$ and the probability measures $d\mu, d\nu$ on the set M are equal, the following corollary holds.

Let a linear closed topological subspace $S_p^{(q)} \subset L_p(M, d\mu)$, $p > 1$ ($\neq 2$), be identically embedded into a Banach space $L_q(M, d\nu)$ for $q = 2 + (p - 2)2^m > p > 1$ ($\neq 2$), where $d\mu$ is a probability measure on M. Then the dimension $\dim S_p^{(q)} = N \in \mathbb{N}$ of the closed subspace $S_p^{(q)} \subset L_p(M, d\mu)$ proves to satisfy the inequality

$$N \left(\frac{\Gamma\left(\frac{N}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{\sqrt{\pi}\, \Gamma\left(\frac{N+q}{2}\right)} \right)^{2/q} \leq K^2_{p,q(0,m)}$$

for some fixed constant $K_{p,q(0,m)} > 0$. If the degree $q \to \infty$, the dimension $\dim S_p^{(\infty)} \leq K^2_{p,q(0,\infty)}$, being *a priori* finite.

As a consequence of the obtained results for a closed subspace in the Banach space $C([0, 1]; \mathbb{R})$ of continuous functions, closely embedded into $L_q(0, 1; d\mu)$, $q > 1$, we have stated the following Grothendieck-type proposition.

Proposition 6. *Let* $S_c^{(q)} \subset C([0, 1]; \mathbb{R})$ *be a closed subspace of the Banach space* $(C([0, 1]; \mathbb{R}), \|\cdot\|_\infty)$ *of continuous functions on the interval* $[0, 1] \subset \mathbb{R}_+$, *which allows the identical embedding into a Banach space* $L_q(0, 1; d\nu)$, $q > 1$, *with respect to a probability measure* $d\nu$ *on* $[0, 1]$. *Then the subspace* $S_c^{(q)} \subset C([0, 1]; \mathbb{R})$ *is finite-dimensional.*

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