Comaximal graphs of commutative rings, edge ideals, betti numbers, Hilbert series, and homotopy type

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We study the comaximal graphs of the finite rings $\mathbb{Z}_n$ and explain how their arithmetic structure influences algebraic and topological invariants. The vertices are partitioned into units and nonunits, and this decomposition yields a precise description of adjacency in terms of gcd-classes. Using this structure, the edge ideal of $\Gamma(\mathbb{Z}_n)$ is analyzed through the language of Stanley–Reisner theory and independence complexes. Hochster's formula is then applied to derive information about the graded Betti numbers, especially the linear strand. In important cases, explicit formulas for $\beta_{i,i+1}$ are obtained, and the extremal Betti number is shown to satisfy $\beta_{n-1,n} = \varphi(n)$. Also, we highlight the consequences that $\mathrm{reg}(I(\Gamma(\mathbb{Z}_n))) = 2$ and $\mathrm{pdim}(S/I) = n - 1$. On the enumerative side, the Hilbert series is computed from the independence polynomial, f-vector, and h-polynomial of the associated simplicial complex. A topological interpretation is given via the independence complex, whose homotopy type is a wedge of $\varphi(n)$ copies of circles S^0. This decomposition shows that the nonunit part is contractible, while the units contribute isolated components. As a consequence, the edge ring is Cohen–Macaulay precisely when n is prime.

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